UNITED STATES DISTRICT COURT

DISTRICT OF ARIZONA

In Re Bard IVC Filters Products Liability Litigation

No. MD-15-02641-PHX-DGC

EXHIBIT INDEX

PLAINTIFFS' RESPONSE TO BARD'S MOTION TO EXCLUDE THE OPINIONS OF ROBERT M. McMEEKING, PH.D.

Exhibit 1	McMeeking Deposition 7-6-17 Excerpts
Exhibit 2	Boresi & Sidebottom
Exhibit 3	Dieter & Schmidt
Exhibit 4	Briant Deposition 7-13-17 Excerpts
Exhibit 5	Briant Deposition 8-12-16 Excerpts
Exhibit 6	McMeeking Report 4-7-17 (**FILED UNDER SEAL**)
Exhibit 7	Brauer Deposition 5-23-14 Excerpts
Exhibit 8	K130366
Exhibit 9	K102511
Exhibit 10	K101431
Exhibit 11	K093659
Exhibit 12	K082305
Exhibit 13	K080668
Exhibit 14	K073090
Exhibit 15	K062887

Exhibit 16	K052578
Exhibit 17	K050558
Exhibit 18	K031328
Exhibit 19	K022236
Exhibit 20	BPV-17-01-00007760
Exhibit 21	BPV-17-01-00142912
Exhibit 22	BPVE-01-01081046 (**FILED UNDER SEAL**)
Exhibit 23	Moritz Deposition 7-18-17 Excerpts

EXHIBIT 1

Deposition of:

Robert McMeeking , Ph.D.

July 6, 2017

In the Matter of:

In Re: Bard IVC Filters Products Liability

Veritext Legal Solutions
1075 Peachtree St. NE , Suite 3625
Atlanta, GA, 30309
800.808.4958 | calendar-atl@veritext.com | 770.343.9696

investigations of the behavior in bench tests and they would -- they should have designed bench tests that were more effective at investigating the behavior that you would expect to see in the filter when it was eventually implanted in the -- in patients. And they perhaps should have done animal tests as well, but I will not comment specifically on what those animal tests should have been.

Q And you haven't yourself done any further investigations beyond what Bard did to see if you can determine, through investigation, whether Bard in fact failed to understand or determine some characteristic in their filter that was leading to complications?

MR. O'CONNOR: Object to the form of the question.

THE WITNESS: I -- I have not done such investigations.

BY MS. DALY:

Q And you also have not developed any bench testing that you think would have been better testing that Bard could have done to reveal something about their filter that identified information about these complications?

A I don't do bench testing, so no, I haven't

developed any bench tests.
Q And you just said you hadn't done any animal testing and you didn't --
A No.
Q -- have any ideas of protocols?
A I don't do animal testing and I --
Q Okay. With respect to the comment you just made about Bard should have taken certain measures sooner, we've talked about some of these before. The electropolishing issue, for example, you've testified before that you thought they should have done that earlier?
A I believe I said I rely on prof- -- Dr. Richie in regard to the question of electropolishing the wires that are in the filters.
Q Okay. So with respect to an opinion that Bard could have electropolished its retrievable filters before it did so in the Eclipse, you're not going to give that opinion, you defer to Dr. Richie?
A Well, I --
MR. O'CONNOR: Form and foundation.
THE WITNESS: I'll defer to his opinion in terms of the wires, but a point I would like to make is that they could have switched to using tube

materials sooner, and they could have made the material out of tube material which they could -- which they could have electropolished at the stage of -- of making the filters from tube material rather than wires.

BY MS. DALY:

Q I'm sorry, I'm missing what that word is. What materials?

A Oh, so --

Q You said troop?

A Tube.

Q Tube materials.

A Tube, yeah.

Q Got it.

A T-u-b-e.

Q Okay. Do you know of any manufacturer that was using tube materials to make IVC filters before the time that Bard came out with the electropolished Eclipse?

A No.

Q Do you have any papers you can cite me to that that -- that one could electropolish wire adequately to have it improve any characteristic of an IVC filter before Bard did so in the Eclipse?

MR. O'CONNOR: Form.

THE WITNESS: I've -- I've not investigated that aspect of the situation, and as I said before, I defer to Dr. Richie in regard to electropolishing wires.

BY MS. DALY:

Q Are there any other changes that you think Bard later made to its filters that it could have made earlier --

A Yes.

Q -- to -- to impact resistance to complications?

A Yes.

Q All right. And what are those?

A They could have developed caudal anchors sooner than they ultimately did. They could have developed penetration limiters sooner than they ultimately did. And they could have redesigned the filter configuration to try and find a better -- a better combination of -- of -- of phenomena that would improve the behavior of the filter in terms of the risks involved.

Q All right. So let's talk about caudal anchors and limiters. On what do you base your opinion that Bard could have added caudal anchors and limiters earlier than it did?

Denali to look specifically at shapes, diameters of limbs, numbers of limbs, new materials?
A Well, when they went from the Recovery to the G2, they changed the shapes and the lengths of -- of the limbs. When they went to the G2X, they made some changes to the details of the cap shape. And then of course the next big change -- adding caudal anchors and electropolishing were changes, but the next big change was moving to the Denali where they use a tube instead of wires to design the -- the filter. But I'm not aware of whether they considered other changes such as changing the number of limbs or moving to a different material.
Q Have you done any work to -- to look at combinations of things like type of material, diameter, shapes of limbs, numbers of limbs, that you think would cause an IVC filter to perform better?
A Well, in --
MR. O'CONNOR: Object to the form of the question.
THE WITNESS: In this case, I've done a comparison of the -- of the Recovery through Denali line of filters with the Simon nitinol filter.

fracture known and foreseeable risks of IVC filters?
A Of many IVC filters, yes.
Q And are you aware that some of those events have undesirable effects on the patient and some don't?
MR. O'CONNOR: Form.
THE WITNESS: Well, I'm not a medical expert so I can't really answer that question.
BY MS. DALY:
Q Fair enough.
And similarly, as an -- as an engineer, you're not qualified to give an opinion as to what the benefits are of any -- to any given patient of the use of an IVC filter, true?
MR. O'CONNOR: Form.
THE WITNESS: No, I would not do that. I would not offer such opinions on the benefits.
BY MS. DALY:
Q Okay. Are you giving any opinions in this litigation that Bard complied with or failed to comply with any specific FDA regulations?
A No, I'm not offering such opinions.
Q If you look at page 10 of the report you're looking at now, you say that "Bard was not

frank and honest with the FDA and did not fully inform the FDA of deficiencies in the G2 filter."

Do you see that? It's at the very last long sentence at the bottom of 10.

A Yes. I see that.

Q What's the basis for your opinion?

A The basis, I'm relying on Dr. Parisian for that opinion.

Q Okay. You have not reviewed the materials that she has reviewed, for example, true?

A I've not reviewed --

MR. O'CONNOR: Form.

THE WITNESS: -- all of that material. I probably have seen some of it.

BY MS. DALY:

Q And to your point a moment ago that you are not giving opinions about FDA regulations, is it also fair to say that you are not giving opinions about what Bard's corporate behavior was vis-a-vis what was expected by the FDA?

A I'm --

MR. O'CONNOR: Form.

THE WITNESS: I'm not offering such opinion. Although may I go back and add to my answer to your previous question --

BY MS. DALY:

Q Of course.

A -- which is that in a couple of situations, I've identified information that Bard gave to the FDA which was not correct, was not information that they should have provided as credible information; for example, the claim that the G2 was 12 times better than the Recovery in terms of its fatigue performance and, also, at the same stage when the 510(k) was being undertaken for the G2, they represented calculations that they did that were irrelevant to fatigue as being -- should I stop?

MR. O'CONNOR: No. Finish your answer.

THE WITNESS: Oh.

So they -- they represented calculations that were not done for fatigue situations which were in fact done to estimate the strains upon implantation of the filter into the vena cava. They represented those calculations as ones that could be used to understand or -- or represent fatigue behavior of the filter, and they also represented those calculations as being experimental tests that would validate the fatigue performance of the G2.

So in my view, this information was at the very least misleading.

MR. O'CONNOR: Okay. Hold on a second. Excuse me. What was ringing?

THE COURT REPORTER: Sorry.

MR. O'CONNOR: Thank you.

BY MS. DALY:

Q So I appreciate that answer. My question is: Are you going to take it the next step and give the opinion that the intention of Bard was to be not frank and not honest with the FDA?

MR. O'CONNOR: Form.

THE WITNESS: I -- I will not interpret those actions in that way but, instead, rely on Dr. Parisian for the overall assessment of that situation.

BY MS. DALY:

Q All right. Thank you.

And I guess one other question I have there is: Do you have any idea what the FDA did with the information that you've just described for us that Bard gave them about the G2? Do you know what they -- if they relied on it, if they asked questions about it, if they rejected it? Do you know?

MR. O'CONNOR: Form.
THE WITNESS: Well, I know that because of some reports of contacts, reports with -- with the FDA, that these points were -- well, the matter of the 12 times better fatigue performance was -- was discussed with the FDA.
BY MS. DALY:
Q Okay. Your report that we're on now, back at page 6, Section 3.2.2.2, it's entitled "FDA Mandated Quality System Regulation Design Controls." And you just -- you cite to some of those regs, correct?
A Correct.
Q And then in 3.2 -- 3.2.2.1 you talk about the class of devices that those -- the design controls relate to, correct?
A Correct.
Q In citing this section or including this section, are you providing any opinion that Bard failed to comply with FDA regulations as they relate to design controls, quality systems, design validation, design output or verification that are listed on pages 6 and 7 of your report?
MR. O'CONNOR: Form.
THE WITNESS: No, I'm not offering any

limited.

Q Would -- do you have an opinion whether those anchors or limiters on the Meridian would add fracture resistance to that filter?

A I have no opinion on that.

Q Same questions with Denali, do you think that the limiters that the Denali has will act to improve resistance to migration, tilt, perforation and fracture?

MR. O'CONNOR: Form.

THE WITNESS: It's -- it is reasonable to expect that there will be some effect on -- on tilt and migration and that those would have possible knock-on consequences to perforation and fracture. And so I'd like to revise my answer about the Meridian in the same way, that the caudal anchors, to the extent they limit tilt and migration, they could have beneficial effects on perforation and fracture.

BY MS. DALY:

Q Okay. What modifications to the G2 filter assisted in resistance to cephalic migration? Do you have an opinion on that?

MR. O'CONNOR: Form.

THE WITNESS: I'm not aware of any changes

A I have no opinion on that because I have not studied it.

Q Okay. Have you studied what the mechanism is anatomically to create caudal migration in any Bard filter?

A I've not studied that independently.

Q Okay.

A I've only --

Q What does that mean?

A What does it mean. I've looked at the Bard report of their bench test --

Q Okay.

A -- and -- which is suggestive of a mechanism that can drive caudal migration.

Q Okay.

A Because it's -- because it's associated with tilt.

Q Okay. And have you done any work to determine how Bard might have modified its filters to reduce tilt that you associate with caudal migration, with contributing to caudal migration?

MR. O'CONNOR: Form.

THE WITNESS: Well, the only observation I have is that the effective caudal anchors would have had a beneficial effect, but otherwise I've

done no thinking or studying of that.

BY MS. DALY:

Q All right. If you look at your report, page 13, it's the G2 Express filter.

A Yes.

Q We've talked about the cap change I think exhaustively.

A Yes.

Q And did you have any other observation of the G2 Express as -- as having characteristics that that particular filter had that contributed to any of these complications different from Recovery and G2?

A Well, the cap -- sorry, the hook on the cap, because it would have touched -- during tilt it would have been touched, under certain assumptions about how the filth occurs, it would have touched the wall of the vena cava first as opposed to other points on the cap, and that would have had some effect on what happens in terms of perforation and tilting of the -- of the filter.

And it's my assumption that the big hook would have taken -- would have been -- would have taken longer to perforate through the wall of the vena cava than the cap itself, although that's just

either with your analysis?
A No, I have not.
MR. O'CONNOR: Form.
BY MS. DALY:
Q Okay. Are you aware of any FDA regulations relating to testing that Bard failed to meet?
MR. O'CONNOR: Form.
THE WITNESS: I'm -- I'm not giving any opinion on what they did relative to requirements of the FDA.
BY MS. DALY:
Q All right. Thank you.
Are you going to provide an opinion that Bard had a higher rate of any particular type of complication relative to other filters?
MR. O'CONNOR: Form.
THE WITNESS: I'm not going to offer any opinion on the relative rates of complications of one filter versus another.
BY MS. DALY:
Q What about one Bard filter versus another Bard filter?
A I'm not going to give an opinion on that because I don't have enough data to truly assess

Q Okay. Was it just fracture that you were looking at or that -- that you rely on in any way in the report?

A Well, the -- so there's statistical differences among the filters compared to the Simon nitinol and not just the fracture results but in some other of the negative phenomena as well, but I would have to review the document to -- the report to identify that explicitly.

Q And in what -- in what way does Dr. Betensky's information support any conclusion that -- or opinion that you have in the case?

MR. O'CONNOR: Form and foundation.

THE WITNESS: Well, it -- it confirms and is consistent with my analysis, confirms in the sense that it's consistent with my analysis in that the Recovery, the G2, the Eclipse and the S- -- and the Meridian, and to some extent the Denali, are subject to rates of failure which are greater than the Simon nitinol filter and that that is consistent with my comparative assessment of the Simon nitinol filter compared to the other filters in the Bard line of products.

BY MS. DALY:

Q Right. And we're going to get to the

but does not change its orientation relative to the vena cava wall, then the stiffness will be four times, which means that the force will -- that you apply will be four times that which you got when you did not constrain the rotation.

Q Okay. And that stiffness in the petal area may make folding that back down, once it's in the patient to put it in a sheath for retrieval, more difficult?

A It would mean --

MR. O'CONNOR: Form.

THE WITNESS: It would mean --

MR. O'CONNOR: Foundation.

THE WITNESS: It would mean that you have to pull on the filter with a bigger force relative to the Recovery catheter to put into that Recovery catheter.

BY MS. DALY:

Q And what that might translate into insofar as patient injury, you do -- you have not done an analysis of that?

A I have not done an analysis of that.

Q All right. Now, this is what I wrote down about your conclusions from the SNF report, and I want to talk about these separately and tell me if

I'm missing something.

The first one I wrote down was that it was more resistant to migration than the Bard's retrievable filters. Is that a conclusion that you make?

A Yes, that's correct.

Q Okay. And in what direction is it more resistant to migration?

A Well, my engineering assessment would indicate that it's more resistant to migration after it's been firmly implanted in the vena cava, that it's more resistant to migration in both the caudal and the cephalic direction.

Q Okay. Are you saying that it will never migrate?

A No, I'm not saying that, no.

Q Is it principally the stiffness of the petals that contributes to the migration resistance in this or is it more than that?

A Well, I think it's -- I think it's a combination of the stiffness of the petals and the stiffness of the legs.

Q Is there anything about any other dimensions of the Simon nitinol, diameter of wire, length of -- length of anything, height of the

filter overall, anything like that, that contributes to migration resistance?

A Well, the diameter of the wire controls the stiffness of the wire -- of the components that are made from the wire, so that has a -- that has an effect, which I've already alluded to in the -- in the answers I just gave.

Q Okay.

A The length of the petals and the lengths of the legs also contribute to controlling the stiffness, so those would contribute as well. And I'm not sure if I can identify anything else, but those were -- those would be the things that I would identify.

Q Did you do any analysis of how one would make changes to either the petal dome or the legs of the SNF to allow it to be retrievable?

A Can I augment my answer of just a second ago? The -- the diameter or the span of the petals and the span of the arms -- the legs relative to the diameter of the vena cava would contribute to the forces which are involved and, therefore, contribute to the question of whether migration is or is not likely in the Simon nitinol filter.

But to move on to your subsequent

to tilt more but --
Q In the SNF or the --
A In the SNF.
Q Okay.
A But there is a feature of the design of the SNF which is that you have these two bushings with some compliance in between them which allows the top of the filter to rotate relative to the bottom, which means that -- that while there are high driving forces for the tendency to tilt, there is a more forgiving aspect to the filter that enables it to accommodate the tendency to tilt perhaps by the petals alone tilting but not the legs or the legs alone tilting but not the petals.
Q And what accommodation are you speaking of? Meaning that the tilt doesn't have consequences beyond tilting or that -- what do you mean by that?
A Yeah, I mean that the -- that the tilting -- I mean that the tilting would be self-limiting and the -- the driving force would not be as continuous as it would be in the G2 and Recovery filters, and, therefore, the extent of net tilting of the filter is likely to be less in the case of the Simon than in the Recovery and the G2.

Q Any other reports that were marked today that you prepared?
A That I prepared? I don't believe so, no.
Q All right. Now, Dr. McMeeking, do the reports that you've talked about today, the ones we just listed, do those set forth your opinions and the bases for the opinions that you've arrived at that are to a reasonable degree of engineering probability?
A Yes, they do.
Q And in arriving at your opinions, did you follow a methodology?
A I did.
Q Did you follow a methodology that is utilized by engineers in your field to solve and resolve engineering problems?
A Yes, I used the standard method knowledge that's used by engineers in my field across a broad range of -- of engineering problems, in addition -- and, in addition, that I used specifically in addressing design analysis and testing of medical implants for companies that manufacture such devices.
Q Did you exercise the same level of intellectual rigor that's used by engineers,

whether in litigation or outside of litigation, to -- to address engineering issues and engineering problems?

A I did.

Q Let's just talk about one of your opinions. Based upon your assessment of the Bard filters, the Bard retrievable filters, the G2, the G2X, the Recovery, the Eclipse and the Denali and Meridian, is it your opinion that failure modes that we've talked about, tilt, perforation, migration and fracture, are predictable based upon those designs?

A Yes, they are.

Q Is that an opinion that you hold to a reasonable degree of engineering probability?

A I do.

Q Now, in your opinions, you talked about calculations that you performed?

A Yes.

Q Was it necessary to perform calculations on each of the model filters or were you able to use calculations that you have arrived at and apply those to the Bard filter and Bard filter models?

A I was able to do calculations for certain models and then make use of the results of those

calculations to understand and make a -- come to opinions on the other models that I did not do specific calculations for.

Q And could you tell us, you described some through, but tell us the types of calculations, the types of analyses, that you performed in arriving at your opinions in this case.

A Well, I did Euler-Bernoulli beam calculations to look at the strains induced in the limbs of filters as the vena cava expands and contracts. I looked at some finite element calculations for the same problem. I carried out finite element calculations to look at the phenomenon of tilt in -- of the filter in the vena cava.

Q To arrive at the opinions, was it necessary for you to engage in any type of testing such as bench testing?

A No, it wasn't necessary for me to carry out any bench testing.

Q Why?

A Because I had information that was available to me from tests carried out by Bard and, in addition, I had my engineering analysis which enabled me to assess the phenomenon that would take

place in the filters in the circumstances that I described.

Q Your report -- and you were asked some questions today about Bard testing and Bard analyses, so let's just talk about testing. Have you reviewed the testing, bench testing, that Bard did of its filters?

A I've reviewed a lot of the bench testing that Bard did of the filters.

Q And what is your opinion about the bench -- the testing, including bench testing, that Bard did?

A That the testing that they undertook was inadequate and that it was not at a level that would enable them to understand the failures that are likely to occur in the filter.

Q Explain your opinion, if you will.

A Well, for example, they didn't identify the worst-case conditions that the filter would experience, and they didn't test the filter in conditions that would reproduce such worst-case conditions.

Q Do your reports detail your opinions about the inadequacy of testing?

A Yes, they do.

Q And when you talk about worst-case scenario, is that a standard that engineers are required to follow?
A Yes, in dealing with this kind of problem, engineers are expected and required to identify the worst-case conditions and then take that into consideration when assessing the performance of their design and the consequences of -- of the design.
Q Now I want to -- I want to move around a little bit. Well, let me ask you about calculations and analyses. You talked about the ones you've performed. In your work in this case, did you look at and review the types of calculations, the engineering analyses, that Bard performed?
A I reviewed finite element calculations that they performed.
Q And do you have an opinion about whether those were sufficient or adequate?
A Almost all of them were inadequate for various reasons.
Q What were the reasons, among the reasons, please?
A Well, some of the reasons were that it was

clear that they were not carried out in a reliable manner and the results looked inconsistent with each other within sets of calculations and comparing some calculations from one set with calculations from another, and that this was not scrutinized in a way that would enable the discrepancies to be understood so that the calculations could be carried out in a reliable and accurate manner.

In addition, the assumptions that went into the calculations were almost always not appropriate for the calculations that were being done, such as the constraints on the motion of the components of the filter and the way that the calculation was carried out to ensure that accurate results were obtained.

Q If the tests -- the analyses that Bard performed were -- did take the step to be reliable and were accurate, as you suggest they were not, do you have an opinion whether Bard would have known that its filters were predictably going to fail?

A Yes.

MS. DALY: Object to form.

THE WITNESS: Those results, when accurately computed, would have revealed that there

were failure modes that would be predictable when the filters were implanted in patients.

BY MR. O'CONNOR:

Q Does that include migration?

A That includes migration.

Q Fracture?

A Fracture.

Q Tilt?

A Tilt.

Q And perforation?

A And perforation.

Q You were asked questions about the Denali filter. You have evaluated the Denali filter?

A I have evaluated it in -- in -- in terms of its features that make it similar and different from other models in the -- in the Bard line of filters.

Q And have you determined that the design of the Denali fil- -- filter will cause it to predictably fail?

A Yes.

Q And what is the based -- what is that opinion based upon?

A Because it is very similar in shape and configuration to the other filters which also --

which themselves are subject to failures, and the analysis that I've done on the various Bard filters indicates that such failure modes will be present in them. And because of the similarities that the Denali has with the ones that I analyzed, one will expect the same kind of failures to occur in the Denali.

Q You discussed your reports do cite references to medical literature and other literature?

A That's correct.

Q And did that literature that you reviewed appear to have consistent findings that were consistent with your opinions in this case?

A Yes, the observations of -- of the medical literature in terms of the failures that were observed in implanted filters were consistent and support the conclusions and opinions that I drew in regard to the behavior of those filters and where my opinions were based on my calculations.

Q For example, you were asked questions about an article written by Dr. Stavropoulous on the Denali filter. Do you remember that?

A That's correct.

Q And in that article, did it talk about

in the high strain location near the cap, but it doesn't necessarily occur exactly at where the wires of the limbs enter the cap. And so the fracture that takes place a little distance away from the edge of the cap will occur whether the arm itself is touching the cap or not in the circumstances that we're describing.

Q By the way, is it necessary for you to look at an exemplar filter to render your opinions in this case?

A No, it's not.

Q Was it necessary for you to look at explanted filters to explain the cause of filter fractures?

A No, it's not.

Q Did you use appropriate methodology and appropriate foundation to arrive at those opinions?

A Yes, I did.

Q Was it necessary for you to do tests and test -- and look at test results in this case?

A It was not necessary for me to do bench tests or any other kind of tests to come to my opinion.

Q Why not?

A Because I based my opinions on the

calculations that I carried out, and that was sufficient for me to form the opinion that I did.

Q All right. And I'm getting close here. You were asked questions about opinions that you have about the Simon nitinol filter, correct?

A Correct.

Q And you have talked about the Simon nitinol in terms of it is better in terms of failure modes than the other Bard filters; is that correct?

A That's correct.

Q Have you seen Bard internal documents that indicate that the Simon nitinol filter was regarded by Bard to be a better filter in terms of failures and failure modes compared to its other models of filters?

A Yes, I've seen copies of e-mails by Dr. C- -- I can't quite say it correctly -- Cierrela, that refer to the Simon nitinol filter as a better filter than other filters in the Bard line.

Q Why were you -- can you explain the reason that you compared the petal of the Simon nitinol filter to the arm of the Recovery and the G2?

A Because they are elements of the

Page 334

I can't comment on that.

BY MS. DALY:

Q You can't comment on the fact that the SNF is not readily percutaneously retrievable?

A No, I meant that I can't comment on the benefits of having a retrievable filter. That's what I meant. Because I'm not a medical expert.

Q Okay. We'll just leave it at the SNF you know is not readily percutaneously retrievable?

A Correct. I agree.

Q All right. Thank you very much.

FURTHER EXAMINATION

BY MR. O'CONNOR:

Q Just to clarify, the calculations you've done in this case and -- and the assessments you've done, you followed the appropriate methodology, correct?

A Correct.

Q Does that methodology require you to do bench testing?

A No.

Q And in terms of Bard's bench testing, your qualifications and what you did, you followed

appropriate methodology to assess their bench testing?

A I did, and I used that methodology when I consult for medical implant companies, assessing their testing and interpreting it and advising them whether it's being done adequately or whether improvements to the testing should be undertaken.

Q And in this case you determined that the testing was inadequate?

A I did.

Q And that it failed to provide information showing that these filters would fail?

A That's correct.

Q Okay. That's all I have.

And that's an opinion you hold to a reasonable degree of engineering certainty?

A I do.

MR. O'CONNOR: Thanks.

THE COURT REPORTER: Do you read and sign?

MR. O'CONNOR: We'll read and sign.

THE WITNESS: I will read and sign.

THE VIDEOGRAPHER: We are off -- excuse me. We are off the record at 1729, and this concludes today's testimony of Robert McMeeking,

EXHIBIT 2

ADVANCED MECHANICS OF MATERIALS

ARTHUR P. BORESI
Professor and Head of Civil Engineering
University of Wyoming at Laramie

OMAR M. SIDEBOTTOM
Professor Emeritus of Theoretical and Applied Mechanics
University of Illinois at Urbana–Champaign

JOHN WILEY & SONS
New York Chichester Brisbane Toronto Singapore

Library of Congress Cataloging in Publication Data:

Boresi, Arthur P. (Arthur Peter), 1924–
Advanced mechanics of materials.

Includes bibliographical references and indexes.
1. Strength of materials. I. Sidebottom, Omar M. (Omar Marion) II. Title.
TA405.B66 1985 620.1'12 84-17246
ISBN 0-471-88392-1

Printed in the United States of America

10 9 8 7 6 5 4 3

CHAPTER 3

FAILURE CRITERIA

To design a structural part or system to perform a given function, the designer must have a clear understanding of the possible ways or modes by which the part or system may fail to perform the function. In other words, the designer must determine possible *modes of failure* of the system and then establish suitable *failure criteria* that accurately predict the various modes of failure. In general, the determination of modes of failure requires extensive knowledge of the response of a structural system to loads. In particular, it requires a comprehensive stress analysis of the system. Since the response of a structural system depends strongly on the material used, the mode of failure depends strongly on the type of material. In turn, the mode of failure of a given material also depends on the manner or the history of loading—for example, slowly, rapidly, repeatedly applied and removed, and repeatedly reversed (for instance cyclically repeated tension and compression), etc. Accordingly, suitable failure criteria must include effects for different materials, different loading procedures, as well as factors that influence the stress distribution (for example, supports and cracks) in the member.

A major part of this book is concerned with (1) stress analysis, (2) the behavior of various materials subjected to stress and strain, and (3) the theoretical determination of the relationship between appropriate values of stress, strain, displacement, loads, number of repetitions of load, etc. and the mode of failure. The specific mode of failure is related to a significant (critical) value of one of the quantities (for example, load, stress, and strain) associated with failure by an appropriate failure criterion. In addition, attention is devoted to the significance of relations between critical values, determined theoretically, and values determined by experiments on, or by experience with, the response of the structural part or system to loads. In particular, the establishment of *factors of safety* is examined. For example, let P_f be a theoretical critical (failure) load associated with a critical value of a significant quantity (say the maximum shearing stress in the member) for a specific mode of failure (say yielding). Let P_w be a safe working load determined on the basis of

CHAPTER 6

NONSYMMETRICAL BENDING OF STRAIGHT BEAMS

6-1 DEFINITION OF SHEAR CENTER IN BENDING. SYMMETRICAL AND NONSYMMETRICAL BENDING

The straight cantilever beam shown in Fig. 6-1.1 has a cross section of arbitrary shape. It is subjected to pure bending by the end couple $\mathbf{M}_0$. Let the origin 0 of the coordinate system (x, y, z) be chosen at the centroid of the beam cross section at the left end of the beam, with the z-axis directed along the centroidal axis of the beam, and the (x, y)-axes taken in the plane of the cross section. Generally, the orientation of the (x, y)-axis is arbitrary. However, we often choose the (x, y)-axes so that the moments of inertia of the cross section I_x, I_y, and I_{xy} are easily calculated, or we may take them to be principal axes (see the Appendix).

The bending moment, which acts at the left end of the beam (Fig. 6-1.1*a*), is represented by the vector $\mathbf{M}_0$ directed perpendicular to a plane that forms an angle ϕ $(0 \leq \phi < \pi)$ taken positive when measured counterclockwise from the x-z-plane as viewed from the positive z-axis. This plane is called *the plane of load* or *the plane of loads*. A more complete description of the plane of loads is given later on in this Article. Consider now a cross section of the beam at distance z from the left end. The free-body diagram of the part of the beam to the left of this section is shown in Fig. 6-1.1*b*. For equilibrium of this part of the beam, a moment $\mathbf{M}$, equal in magnitude but opposite in sense to $\mathbf{M}_0$ must act at section z. For the case shown $(\pi/2 < \phi < \pi)$, the (x, y)-components (M_x, M_y) of $\mathbf{M}$ are related to the signed magnitude M of $\mathbf{M}$ by the relations $M_x = M \sin\phi$, $M_y = -M\cos\phi$. Since $\pi/2 < \phi < \pi$, $\sin\phi$ is positive and $\cos\phi$ is negative. Since (M_x, M_y) are positive (Fig. 6-1.1*b*), the sign of M is positive. A more complete discussion of the sign convention for M is given in Art. 6-2, following Eq. (6-2.10).

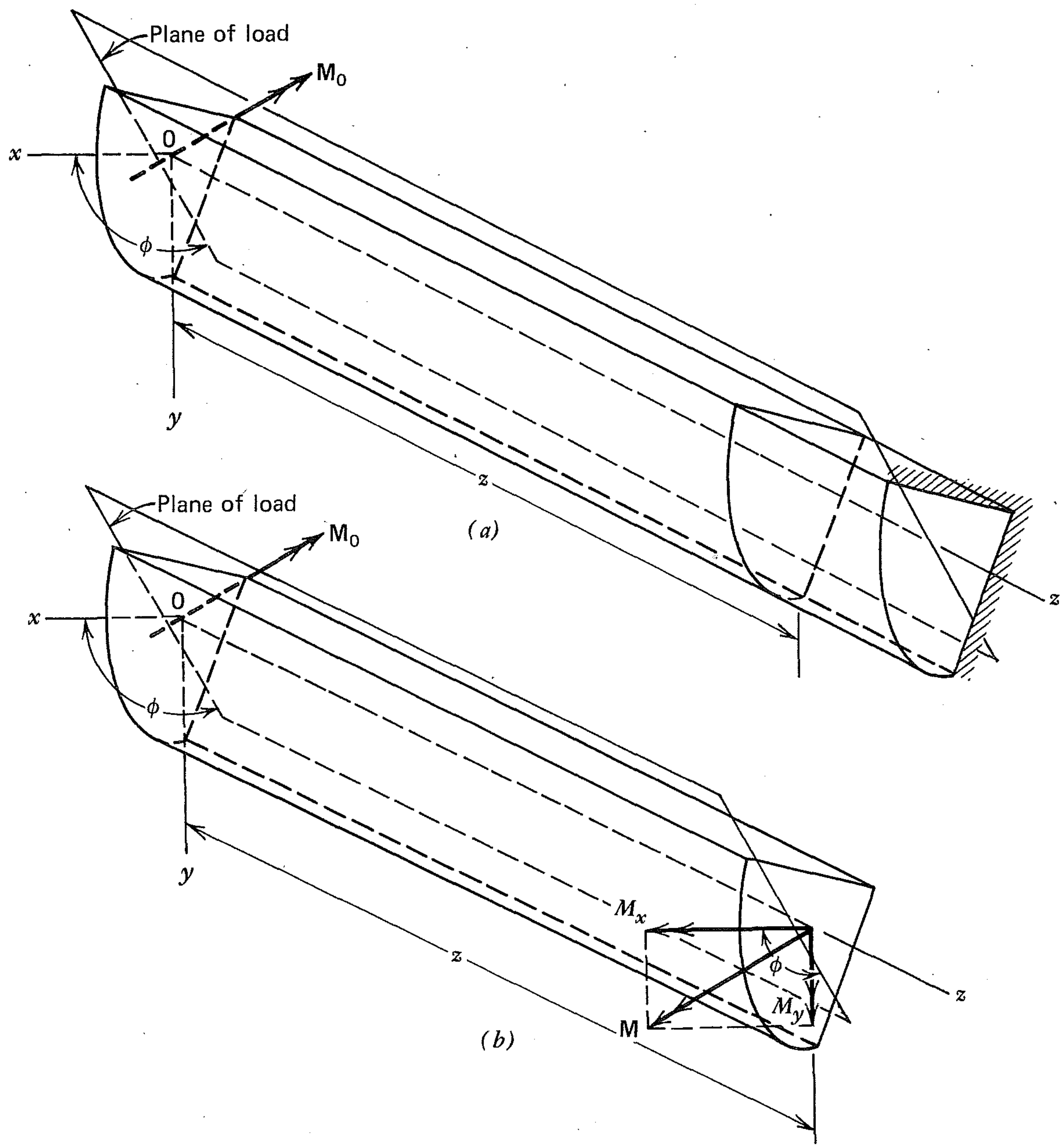


Fig. 6-1.1/Cantilever beam with an arbitrary cross section subjected to pure bending.

Shear Loading of a Beam. Shear Center Defined/Let the beam shown in Fig. 6-1.2*a* be subjected to a concentrated force **P** that lies in the end plane ($z = 0$) of the beam cross section. The vector representing **P** lies in a plane which forms angle ϕ ($0 \leq \phi \leq \pi$), taken positive when measured counterclockwise from the z-x-plane as viewed from the positive z-axis. This plane is called *the plane of the load*. Consider a cross section of the beam at distance z from the left end. The free body diagram of the part of the beam to the left of this section is shown in Fig. 6-1.2*b*. For equilibrium of this part of the beam, a moment **M**, with components M_x and M_y, shear components V_x and V_y, and in general, a twisting moment **T** (with vector directed along the positive z-axis) must act on the section

at z. However, if the line of action of force **P** passes through a certain point C (the shear center) in the cross section, $\mathbf{T}=0$. In this discussion we assume that the line of action of **P** passes through the shear center. Hence **T** is not shown in Fig. 6-1.2*b*. Note that in Fig. 6-1.2*b*, the force **P** requires V_x, V_y to be positive (directed along positive (x, y)-axes, respectively). The component M_x is also directed along the positive x-axis. However, since $\phi < \pi/2$, M_y is negative (directed along the negative y-axis).

There is a particular axial line in the beam called the *bending axis of the beam*, which is parallel to the centroidal axis of the beam (the line

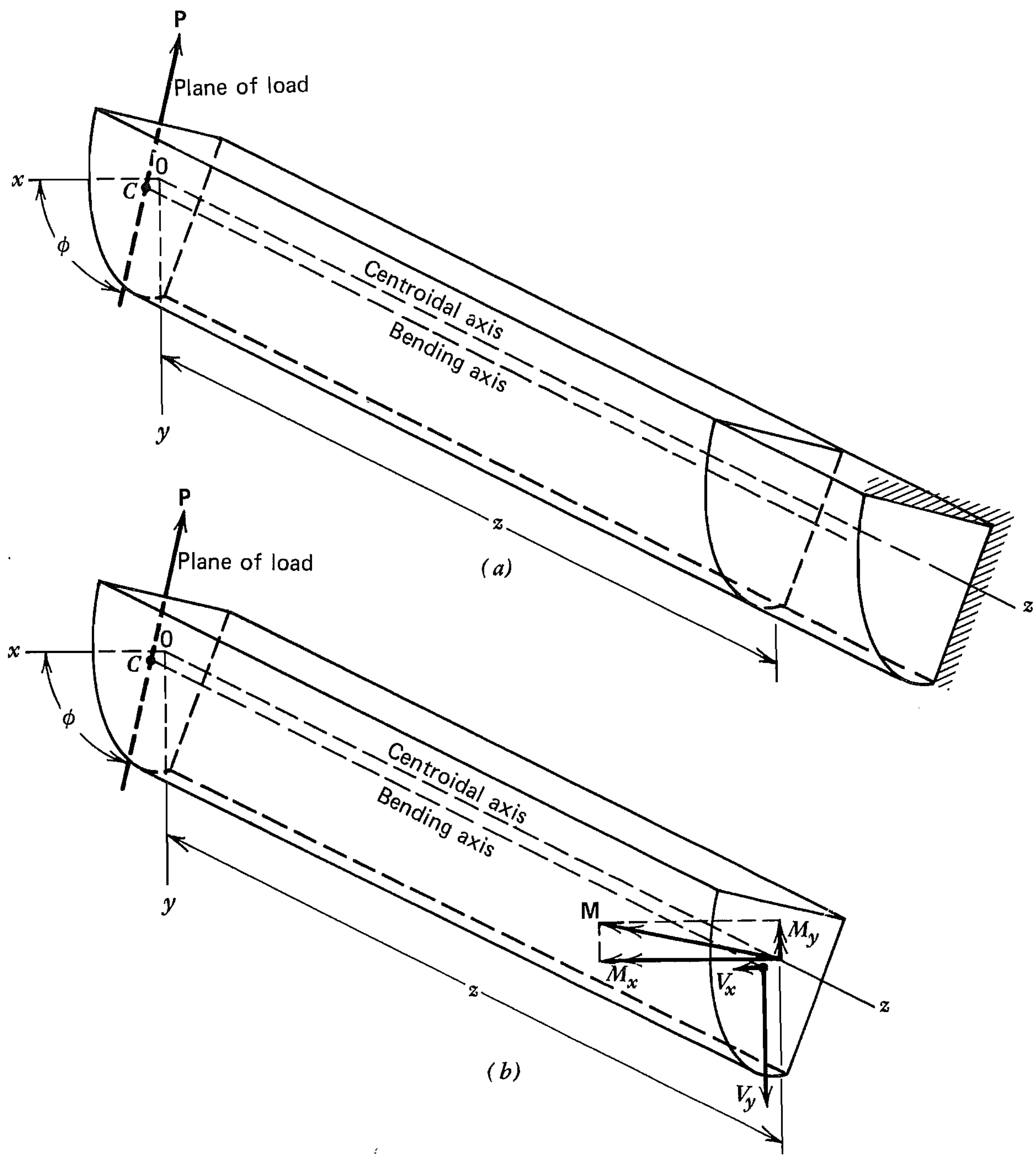


Fig. 6-1.2/Cantilever beam with an arbitrary cross section subjected to shear loading.

which passes through the centroids of all of the cross sections of the beam). Except for special cases, the bending axis does not coincide with the centroidal axis (Fig. 6-1.2).

The intersection of the bending axis with any cross section of the beam locates a point C in that cross section called *the shear center* of the cross section (see Art. 7-1). Thus, the bending axis passes through the shear centers of all the cross sections of the beam.

In Art. 6-2, formulas are derived for the normal stress component σ_{zz} that acts on the cross section at z in terms of the bending moment components (M_x, M_y). Also, one may derive formulas for the shearing stress components (τ_{zx}, τ_{zy}) due to the shearing forces (V_x, V_y). However, if the length L of the beam is large compared to the maximum cross section dimension D, such that $L/D > 5$, the maximum shearing stress is small compared to the maximum normal stress. Hence, in this chapter we ignore the shearing stresses due to (V_x, V_y); that is, we consider beams for which $L/D > 5$. Thus for bending of a beam by a concentrated force and for which the shearing stresses are negligible, the line of action of the force must pass through the shear center of a cross section of the beam; otherwise, the beam will be subjected to both bending and torsion (twist). Thus, the theory of pure bending of beams assures that the shearing stresses due to concentrated loads are negligible and that the lines of actions of concentrated forces that act on the beam pass through the shear center of a beam cross section. If the cross section of a beam has either an axis of symmetry or an axis of antisymmetry, it may be shown that the shear center C is located on that axis (Fig. 6-1.3). If the cross section has two or more axes of symmetry or antisymmetry, the shear center is located at the intersection of the axes (Figs. 6-1.3a and d). For a general cross section (Fig. 6-1.1) or for a relatively thick, solid cross section (Fig. 6-1.3c), the determination of the location of the shear center requires advanced computational methods.[1] For this reason, the location of the shear center is often determined in an approximate manner; the errors introduced by such approximations of the shear center location are discussed in the next paragraph.

Let the line of action of force **P**, Fig. 6-1.2, pass through an approximate location of the shear center of the beam, point B in Fig. 6-1.4. Let C be the location of the shear center. Since the line of action of force **P** does not pass through C, the force **P** is assumed to be replaceable by a couple (torque) that lies in the cross section and a force with an action line that passes through C. This representation or transformation of force **P** is assumed to be valid for the deformable beam cross section, although strictly speaking, it is applicable to rigid bodies only.[2] The transformation is accomplished by adding self-equilibrating forces **P**$'$ and **P**$''$ at C which are parallel to **P** and which have magnitudes equal to that of **P**. Thus, the force **P** is considered to be equivalent to a torque (couple) of magnitude $T = Pd$, due to forces **P** and **P**$''$, where d is the perpendicular distance between **P** and **P**$''$ and a force **P**$'$ acting at C.

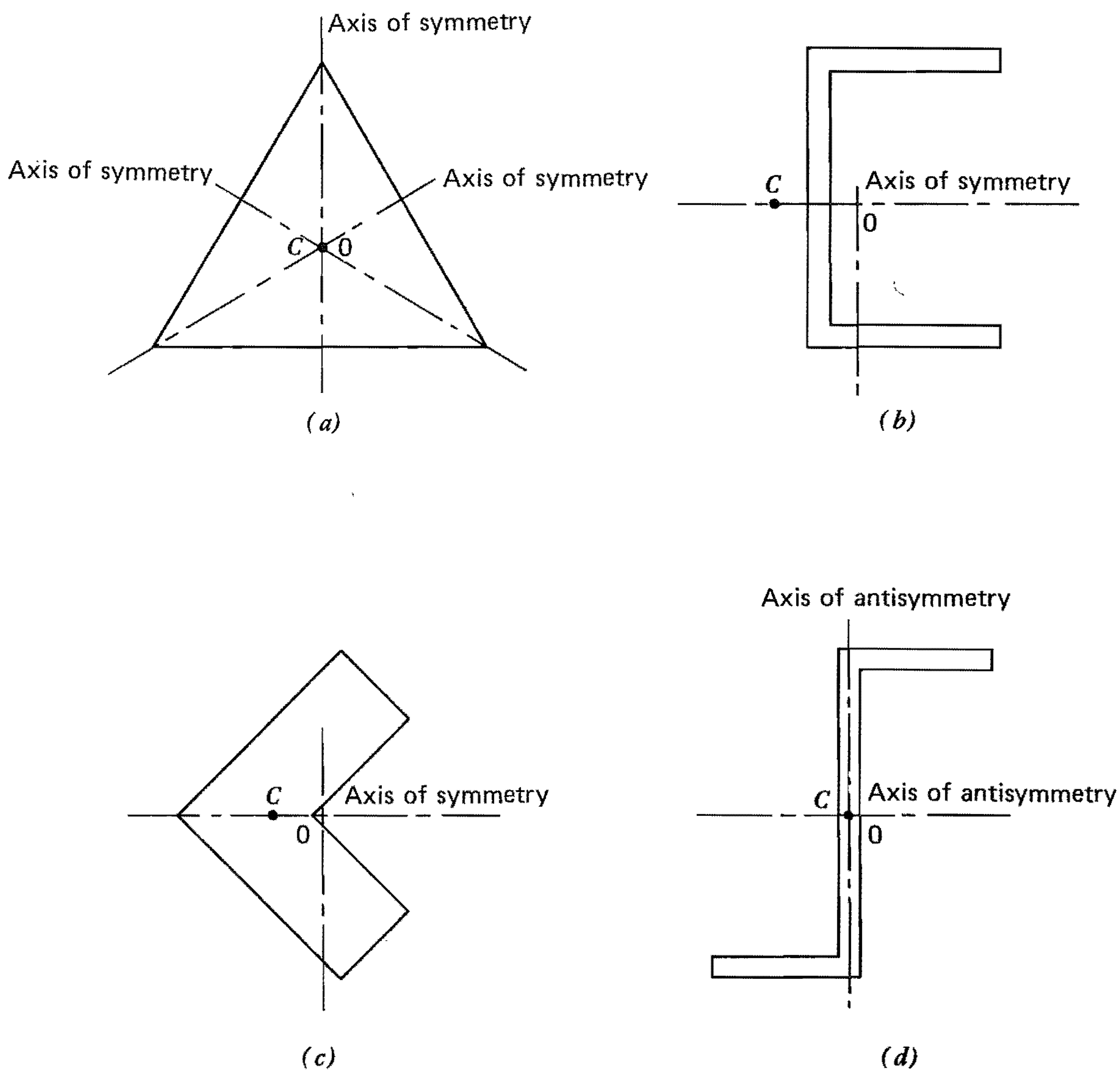


Fig. 6-1.3/(a) Equilateral triangle section. (b) Open channel section. (c) Angle section. (d) Z-section.

Now pass a cutting plane through the member at distance z from the left end. The free body diagram of the beam to the left of the cut is shown in Fig. 6-1.4b. For equilibrium, the forces at the cut include a bending moment **M** with components M_x and M_y, torque of magnitude $T = Pd$, and shears V_x and V_y. The normal stress distribution σ_{zz} due to M_x and M_y can be calculated by the formulas derived in Art. 6-2. The shearing stresses due to V_x and V_y are considered to be negligible ($L/D > 5$). The shearing stress due to torque T may be computed by the methods presented in Chapter 5. Cross sections with thick walls, Fig. 6-1.3c, require large torques if the maximum shearing stress due to the torque is to be significant. For such cross sections, an approximate location of the shear center will suffice, since shearing stresses due to T are small compared to the maximum value of σ_{zz}, provided Pz is large compared to Pd. However, caution must be used for cross sections made

of connected narrow rectangular walls such as the open channel cross section shown in Fig. 6-1.3*b*, since, as noted in Chapter 5, such cross sections have little resistance to torsional loads. For these kinds of cross sections, an accurate estimate of the location of the shear center is necessary. Such problems are treated in Chapter 7.

In this chapter, the shear centers for many of the cross sections considered are not known exactly. Consequently, unless the shear center

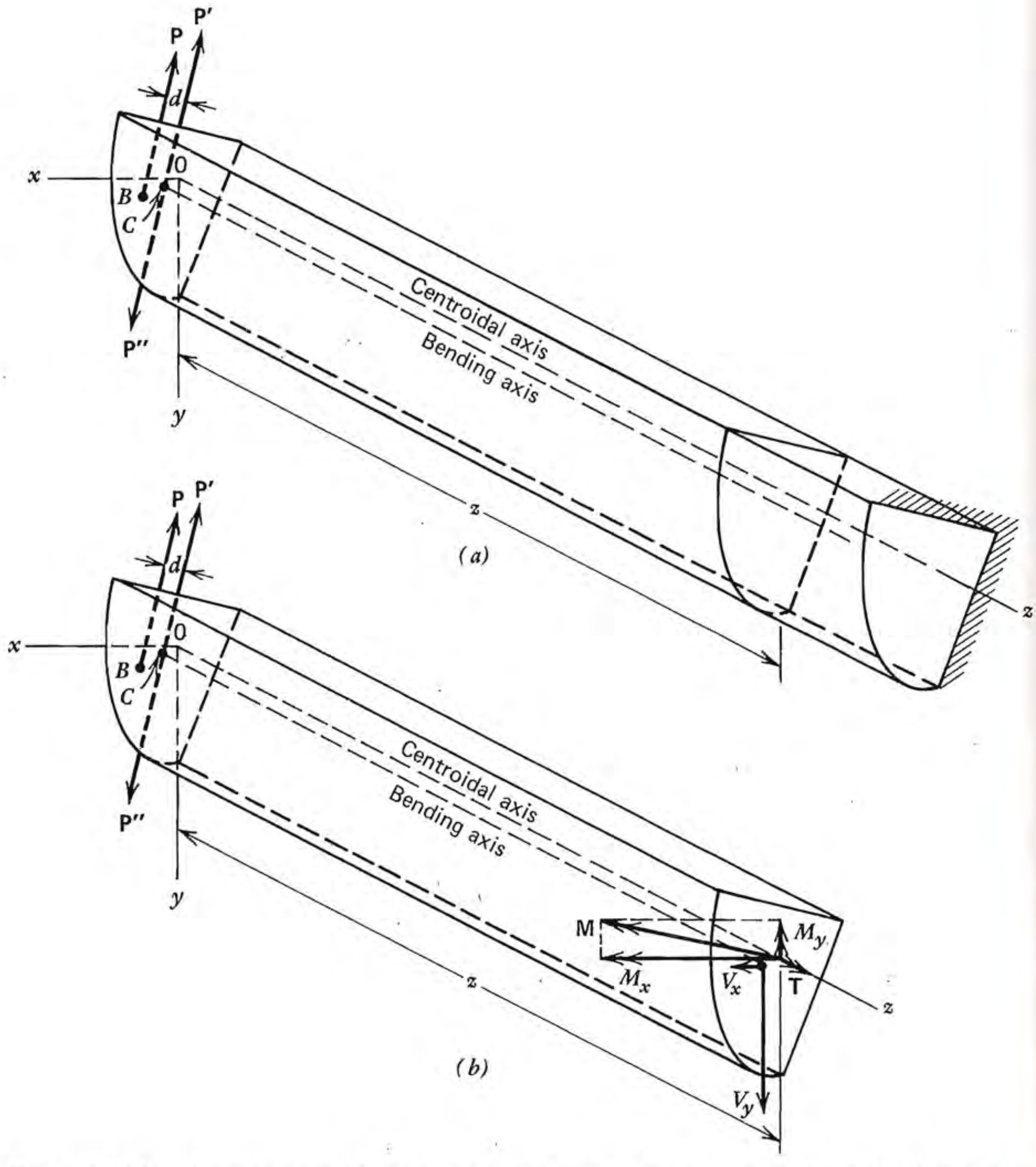


Fig. 6-1.4/Cantilever beam with arbitrary cross section subjected to shear loading not at shear center.

is located by intersecting axes of symmetry or antisymmetry, the location of the shear center is approximated. The reader should have a better understanding of such approximations after studying Chapter 7.

Symmetrical Bending. Nonsymmetrical Bending/In the Appendix, it is shown that every beam cross section has principal axes (X, Y). With respect to principal axes (X, Y), the product of inertia of the cross section is zero; that is, $I_{XY} = 0$. The principal axes (X, Y) for the cross section of the cantilever beam of Fig. 6-1.1 are shown in Fig. 6-1.5. For convenience, axes (X, Y) are also shown at the section distance z from the left end of the beam. At the left end, let the beam be subjected to a couple $\mathbf{M}_0$ with sense in the negative X direction and a force $\mathbf{P}$ at the shear center C with sense in the negative Y direction (Fig. 6-1.5*a*). These loads produce a bending moment $\mathbf{M} = \mathbf{M}_X$ at the cut section with sense in the positive X direction. By Bernoulli beam theory,[1] the stress σ_{zz} normal to the cross section is given by the flexure formula

$$\sigma_{zz} = \frac{M_X Y}{I_X} \qquad (6\text{-}1.1)$$

where Y is the distance from the principal axis X to the point in the cross section at which σ_{zz} acts, and I_X is the principal moment of inertia of the cross-sectional area relative to the X-axis. Equation (6-1.1) shows that σ_{zz} is zero for $Y = 0$ (the X-axis). Consequently, the X-axis is called the *neutral axis of bending of the cross section*; that is, the axis for which $\sigma_{zz} = 0$. We define the bending moment component M_X as positive when the sense of the vector representing $\mathbf{M}_X$ is in the positive X direction. Since M_X is related to σ_{zz} by Eq. (6-1.1), σ_{zz} is a tensile stress for positive values of Y and a compressive stress for negative values of Y. In addition to causing a bending moment component M_X, load P produces a positive shear V_Y at the cut section. It is assumed that the maximum shearing stress τ_{ZY} resulting from V_Y is small compared to the maximum value of σ_{zz}. Hence, since this chapter treats pure bending effects, we neglect shearing stresses in this chapter.

Likewise, if a load $\mathbf{Q}$ (applied at the shear center C), directed along the positive X-axis, and a moment $\mathbf{M}_0$ directed along the negative Y-axis, are applied to the left end of the beam (Fig. 6-1.5*b*), they produce a bending moment $\mathbf{M} = \mathbf{M}_Y$ directed along the positive Y-axis. The normal stress distribution σ_{zz}, due to the component M_Y, is also given by the flexure formula. Thus,

$$\sigma_{zz} = -\frac{M_Y X}{I_Y} \qquad (6\text{-}1.2)$$

where X is the distance from the principal axis Y to the point in the cross section at which σ_{zz} acts, and I_Y is the principal moment of inertia of the

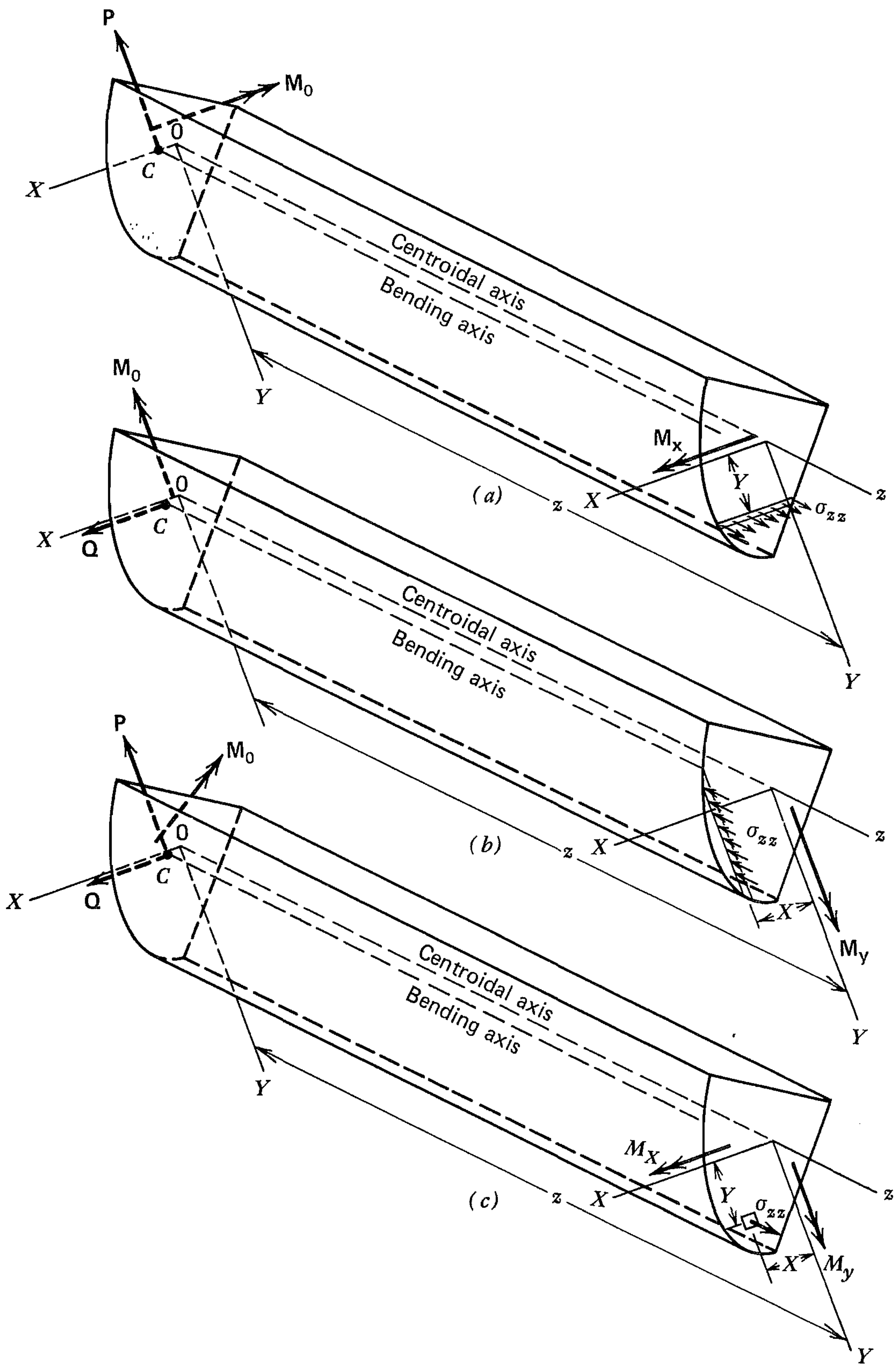


Fig. 6-1.5/Cantilever beam with an arbitrary cross section.

cross-sectional area relative to the Y axis. The negative sign arises from the fact that a positive M_Y produces compressive stresses on the positive side of the X axis. Now for $X = 0$ (the Y-axis), $\sigma_{zz} = 0$. Hence, in this case, the Y-axis is the *neutral axis of bending of the cross section*; that is, the axis for which $\sigma_{zz} = 0$. In either case (Eq. (6-1.1) or (6-1.2)), the beam is subjected to *symmetrical bending*. (Bending occurs about a neutral axis in the cross section that coincides with the corresponding principal axis.)

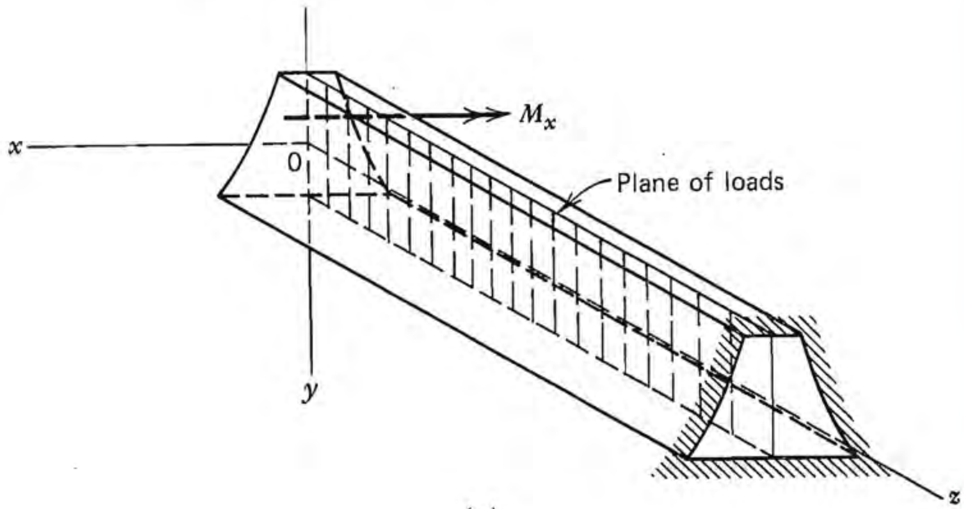
In Fig. 6-1.5*c*, the beam is subjected to moment $\mathbf{M}_0$ with components in the negative directions of both axis (X, Y), as well as concentrated forces **P** and **Q** acting through the shear center C. These loads result in a bending moment **M** at the cut section with positive components (M_X, M_Y). For this loading, the stress σ_{zz} normal to the cross section may be obtained by superposition of Eqs. (6-1.1) and (6-1.2). Thus,

$$\sigma_{zz} = \frac{M_X Y}{I_X} - \frac{M_Y X}{I_Y} \tag{6-1.3}$$

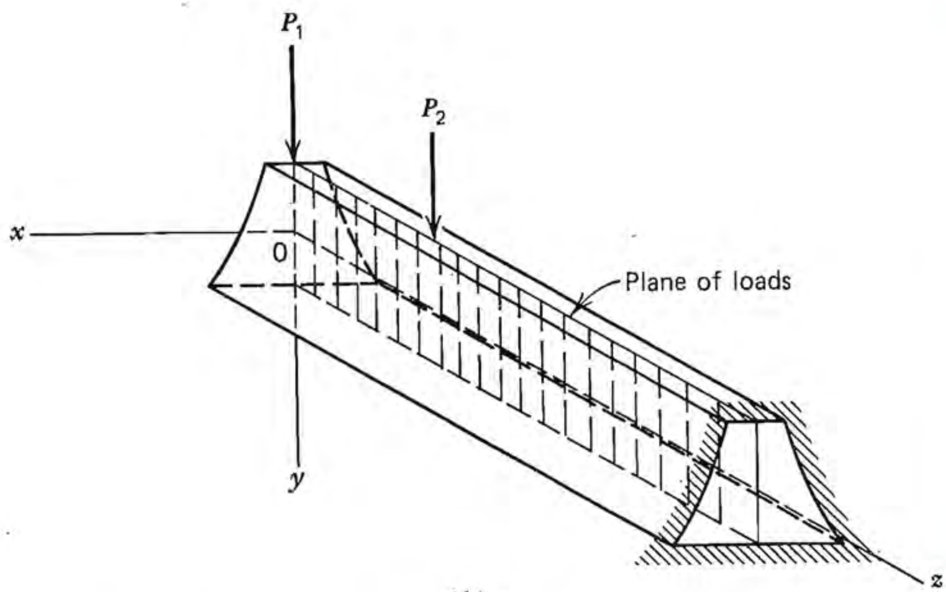
In this case, the moment $\mathbf{M} = (M_X, M_Y)$ is not parallel to either of the principal axis (X, Y). Hence, the bending of the beam occurs about an axis that is not parallel to either the X- or Y-axis. When the axis of bending does not coincide with a principal axes direction, the bending of the beam is said to be nonsymmetrical. The determination of the *neutral axis of the cross section* for nonsymmetrical bending is discussed in Art. 6-2.

Plane of Loads. Symmetrical and Nonsymmetrical Loading/Often a beam is loaded by forces that lie in a plane which coincides with a plane of symmetry of the beam, Fig. 6-1.6. Since, then, the y-axis is an axis of symmetry for the cross section, it is a principal axis. Hence, if axes (x, y) are principal axes for the cross section, the beams in Figs. 6-1.6*a* and *b* undergo symmetrical bending; that is, bending about a principal axis of a cross section, since the moment vector in Fig. 6-1.6*a* and the force vectors in Fig. 6-1.6*b* are parallel to principal axes. (See the discussion above in the section entitled "Symmetrical Bending. Nonsymmetrical Bending.") We further observe that since the shear center lies on the y-axis, the plane of the load contains the axis of bending of the beam. More generally, it is shown later in this chapter that if the plane of loads does not coincide with a plane of symmetry of the beam, the beam may still deform symmetrically (bend about a principal axis), provided that the plane of loads contains the bending axis and is parallel to one of the principal planes [the (x, z)- and (y, z)-planes in Fig. 6-1.6].

Consider next two beams with cross sections shown in Fig. 6-1.7. Since a rectangular cross section (Fig. 6-1.7*a*) has two axes of symmetry that pass through its centroid 0, the shear center C of a rectangular cross section (which is located at the intersection of the two axes of symmetry)



(a)



(b)

Fig. 6-1.6/Plane of loads coincident with the plane of symmetry of the beam. (a) Couple loads. (b) Lateral loads.

coincides with the centroid 0. Let the intersection of the plane of the loads and the plane of the cross section be denoted by line L-L, which forms angle ϕ ($0 \le \phi < \pi$) measured counterclockwise from the x-z-plane, and which passes through the shear center C. Since the plane of loads contains point C, the bending axis of the rectangular beam lies in the plane of the loads. If the angle ϕ equals 0 rad or $\pi/2$ rad, the rectangular beam will undergo symmetrical bending; that is, bending about a principal axis. For other values of ϕ, the beam undergoes nonsymmetrical bending; that is, bending for which the neutral axis of bending of the cross section does not coincide with either of the principal axes x, y.

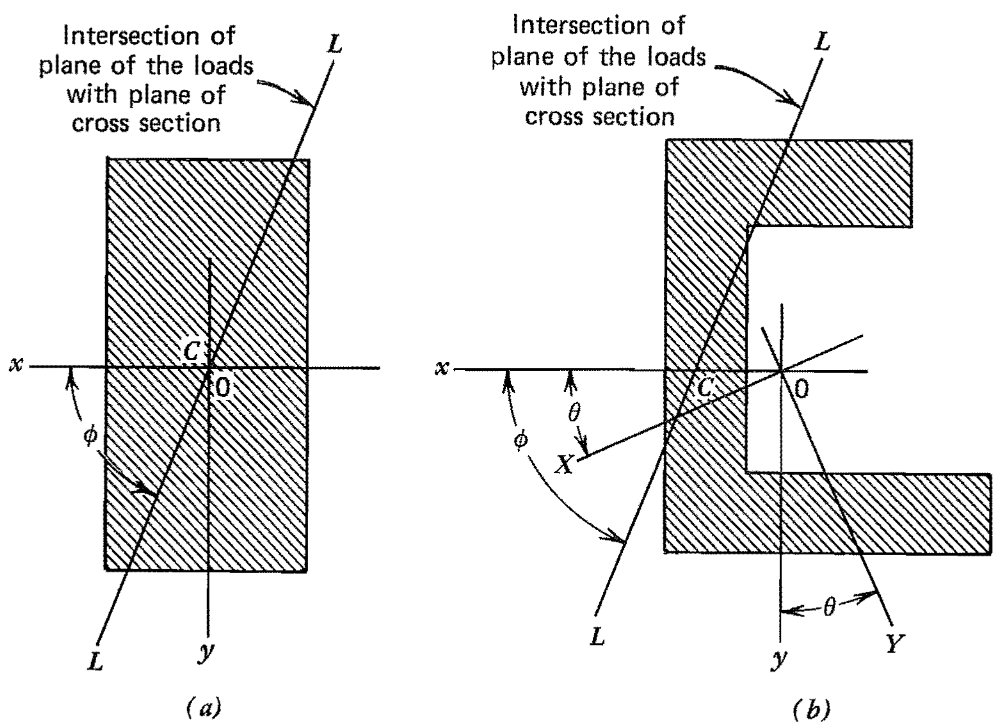


Fig. 6-1.7/Unsymmetrically loaded beams. (a) Rectangular cross sections. (b) Channel cross section.

In the case of a general channel section (Fig. 6-1.7*b*), the principal axes (X, Y) are located by a rotation through angle θ (positive θ is taken counterclockwise) from the (x, y)-axes as shown. The value of θ is determined by Eq. (A-3.4), in the Appendix. Although the plane of loads contains the shear center C (and, hence, the bending axis of the beam), it is not parallel to either of the principal planes (X, z), (Y, z). Hence, in general, the channel beam (Fig. 6-1.7*b*) undergoes nonsymmetrical bending; that is, bending about an axis that is not a principal axis. However, for the two special cases, $\phi = \theta$ or $\phi = \theta + \pi/2$, the channel beam undergoes symmetrical bending.

6-2 BENDING STRESSES IN BEAMS SUBJECTED TO NONSYMMETRICAL BENDING

Let a cutting plane be passed through a straight cantilever beam at section z. The free body diagram of the beam to the left of the cut is shown in Fig. 6-2.1*a*. The beam has constant cross section of arbitrary shape. The origin 0 of the coordinate axes is chosen at the centroid of the beam cross section at the left end of the beam with the z-axis taken parallel to the beam. The left end of the beam is subjected to a bending couple $\mathbf{M}_0$ which is equilibrated by bending moment $\mathbf{M}$ acting on the

cross section at z, with positive components (M_x, M_y) as shown. The bending moment $\mathbf{M} = (M_x, M_y)$ is the resultant of the forces due to the normal stress σ_{zz} acting on the section (Fig. 6-2.1b). For convenience, we show (x, y)-axes at the cross section z. It is assumed that the (x, y)-axes are not principal axes for the cross section. In this article, we derive the load-stress formula which relates the normal stress σ_{zz} acting on the cross section to the components (M_x, M_y).

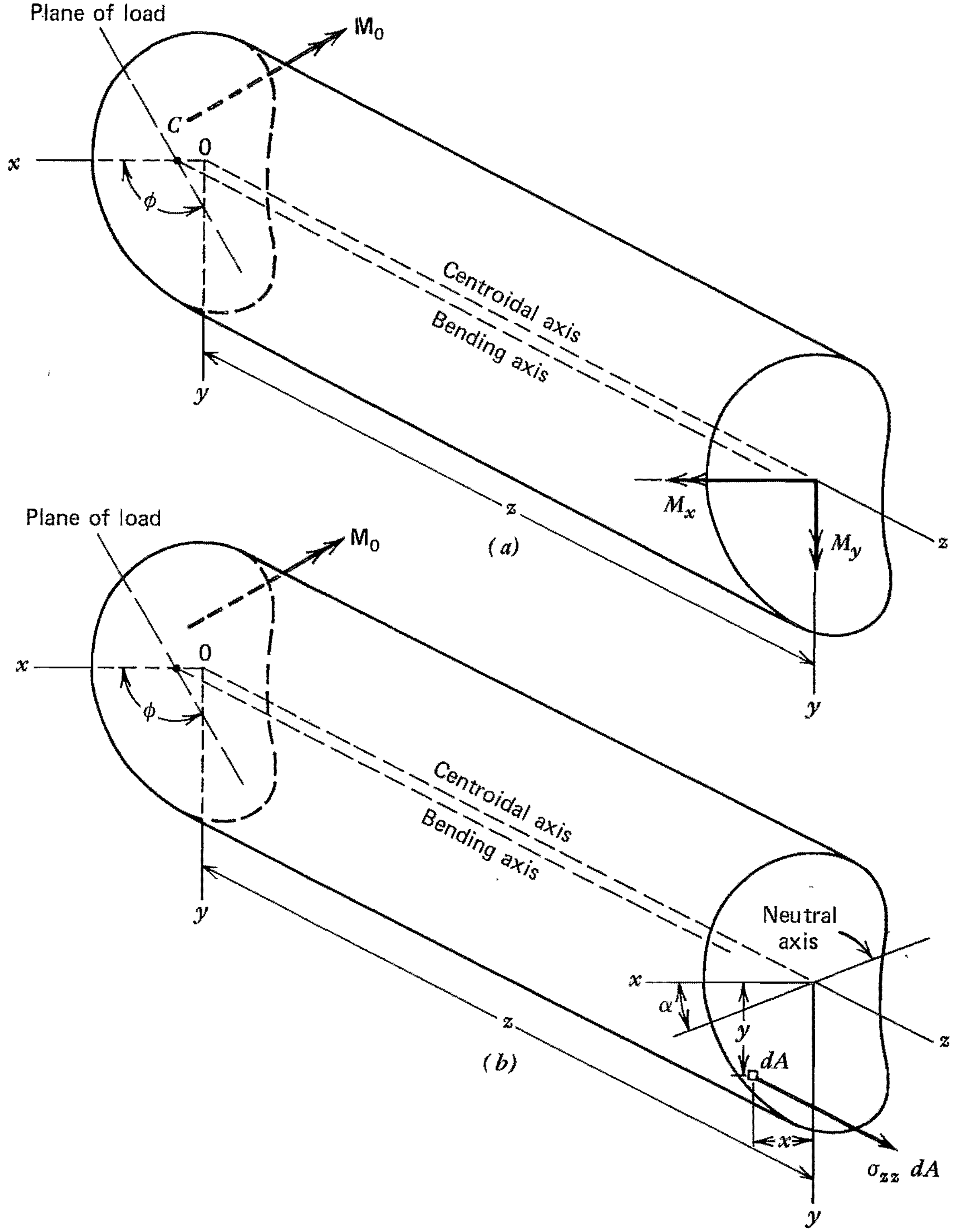


Fig. 6-2.1/Pure bending of a nonsymmetrically loaded cantilever beam.

As discussed in the introduction to Chapter 2, the derivation of load-stress and load-deformation relations for the beam requires that equations of equilibrium, compatibility conditions, and stress-strain relations be satisfied for the beam along with specified boundary conditions for the beam.

Equations of Equilibrium/Application of the equations of equilibrium to the free body diagram in Fig. 6-2.1*b* yields (since there is no net resultant force in the z direction)

$$\begin{aligned} 0 &= \int \sigma_{zz}\, dA \\ M_x &= \int y\sigma_{zz}\, dA \\ M_y &= -\int x\sigma_{zz}\, dA \end{aligned} \qquad (6\text{-}2.1)$$

where dA denotes an element of area in the cross section and the integration is performed over the area A of the cross section. To evaluate the integrals in Eq. (6-2.1), it is necessary that the functional relation between σ_{zz} and (x, y) be known. The determination of σ_{zz} as a function of (x, y) is achieved by considering the geometry of deformation and the stress-strain relations.

Geometry of Deformation/We assume that plane sections of an unloaded beam remain plane after the beam is subjected to pure bending. Consider two plane cross sections perpendicular to the bending axis of an unloaded beam such that the centroids of the two sections are separated by a distance Δz. These two planes are parallel since the beam is straight. These planes rotate with respect to each other when moments M_x and M_y are applied. Hence, the extension e_{zz} of longitudinal fibers of the beam between the two planes can be represented as a linear function of (x, y); namely,

$$e_{zz} = a'' + b''x + c''y \qquad (6\text{-}2.2)$$

where a'', b'', and c'' are constants. Since the beam is initially straight, all fibers have the same initial length Δz so that the strain ϵ_{zz} can be obtained by dividing Eq. (6-2.2) by Δz. Thus,

$$\epsilon_{zz} = a' + b'x + c'y \qquad (6\text{-}2.3)$$

where $\epsilon_{zz} = e_{zz}/\Delta z$, $a' = a''/\Delta z$, $b' = b''/\Delta z$, and $c' = c''/\Delta z$.

Stress-Strain Relations/According to the theory of pure bending of straight beams, the only nonzero stress component in the beam is σ_{zz}. For linearly elastic conditions, Hooke's law states

$$\sigma_{zz} = E\epsilon_{zz} \qquad (6\text{-}2.4)$$

Eliminating ϵ_{zz} between Eqs. (6-2.3) and (6-2.4), we obtain

$$\sigma_{zz} = a + bx + cy \tag{6-2.5}$$

where $a = Ea'$, $b = Eb'$, and $c = Ec'$.

Load-Stress Relation for Unsymmetrical Bending/Substitution of Eq. (6-2.5) into Eqs. (6-2.1) yields

$$\begin{aligned} 0 &= \int (a + bx + cy)\, dA = a\int dA + b\int x\, dA + c\int y\, dA \\ M_x &= \int (ay + bxy + cy^2)\, dA = a\int y\, dA + b\int xy\, dA + c\int y^2\, dA \\ M_y &= -\int (ax + bx^2 + cxy)\, dA = -a\int x\, dA - b\int x^2\, dA - c\int xy\, dA \end{aligned} \tag{6-2.6}$$

Since the z-axis passes through the centroid of each cross section of the beam, $\int x\, dA = \int y\, dA = 0$. The other integrals in Eqs. (6-2.6) are defined in the Appendix. Equations (6-2.6) simplify to

$$\begin{aligned} 0 &= aA \\ M_x &= bI_{xy} + cI_x \\ M_y &= -bI_y - cI_{xy} \end{aligned} \tag{6-2.7}$$

where I_x and I_y are the centroidal moments of inertia of the beam cross section with respect to the x- and y-axes, respectively, and I_{xy} is the centroidal product of inertia of the beam cross section. Solving Eqs. (6-2.7) for the constants a, b, and c, we obtain

$$\begin{aligned} a &= 0 \qquad \text{(because } A \neq 0\text{)} \\ b &= -\frac{M_y I_x + M_x I_{xy}}{I_x I_y - I_{xy}^2} \\ c &= \frac{M_x I_y + M_y I_{xy}}{I_x I_y - I_{xy}^2} \end{aligned} \tag{6-2.8}$$

The substitution of Eqs. (6-2.8) into Eq. (6-2.5) gives the normal stress distribution σ_{zz} on a given cross section of a beam subjected to unsymmetrical bending in the form

$$\sigma_{zz} = -\left[\frac{M_y I_x + M_x I_{xy}}{I_x I_y - I_{xy}^2}\right] x + \left[\frac{M_x I_y + M_y I_{xy}}{I_x I_y - I_{xy}^2}\right] y \tag{6-2.9}$$

Equation (6-2.9) is not the most convenient form for the determination of the maximum value of the flexure stress σ_{zz}. Before the location of points of maximum tensile and compressive stresses in the cross section can be determined, it is necessary to locate the neutral axis. For this purpose, it is desirable that the neutral axis location include the angle ϕ between the plane of the loads and the x-axis; ϕ is measured positive counterclockwise (Fig. 6-1.7). The magnitude of ϕ is generally in the neighborhood of $\pi/2$ rad ($0 \le \phi < \pi$). The bending moments M_x and M_y

can be written in terms of ϕ as follows:

$$\begin{aligned} M_x &= M\sin\phi \\ M_y &= -M\cos\phi \end{aligned} \tag{6-2.10}$$

in which M is the signed magnitude of moment **M** at the cut section. The sign of M is positive if the x projection of the vector **M** is positive; it is negative if the x projection of **M** is negative. Since the (x, y)-axes are chosen for the convenience of the one making the calculations, they are chosen so that the magnitude of M_x is not zero. Therefore, by Eqs. (6-2.10),

$$\cot\phi = -\frac{M_y}{M_x} \tag{6-2.11}$$

Neutral Axis/The neutral axis of the cross section of a beam subjected to unsymmetrical bending is defined to be the axis in the cross section for which $\sigma_{zz} = 0$. Thus, by Eq. (6-2.9), the equation of the neutral axis of the cross section is

$$y = \frac{M_x I_{xy} + M_y I_x}{M_x I_y + M_y I_{xy}} x = x\tan\alpha \tag{6-2.12}$$

where α is the angle between the neutral axis of bending and the x-axis; α is measured positive counterclockwise (Fig. 6-2.1), and

$$\tan\alpha = \frac{M_x I_{xy} + M_y I_x}{M_x I_y + M_y I_{xy}} \tag{6-2.13}$$

Since $x = y = 0$ satisfies Eq. (6-2.12), the neutral axis passes through the centroid of the section. The right side of Eq. (6-2.13) can be expressed in terms of the angle ϕ by using Eq. (6-2.11). Thus,

$$\tan\alpha = \frac{I_{xy} - I_x\cot\phi}{I_y - I_{xy}\cot\phi} \tag{6-2.14}$$

More Convenient Form for the Flexure Stress σ_{zz}/Elimination of M_y between Eqs. (6-2.9) and (6-2.13) results in a more convenient form for the normal stress distribution σ_{zz} for beams subjected to unsymmetrical bending; namely

$$\sigma_{zz} = \frac{M_x(y - x\tan\alpha)}{I_x - I_{xy}\tan\alpha} \tag{6-2.15}$$

where $\tan\alpha$ is given by Eq. (6-2.14). Once the neutral axis is located on the cross sections at angle α as indicated in Fig. 6-2.1, points in the cross section where the tensile and compressive flexure stresses are maxima are easily determined. The coordinates of these points can be substituted into Eq. (6-2.15) to determine the magnitudes of these stresses. If M_x is zero, Eq. (6-2.9) may be used instead of Eq. (6-2.15) to determine magnitudes of these stresses, or axes (x, y) may be rotated by $\pi/2$ to obtain new reference axes (x', y').

Note: Equations (6-2.14) and (6-2.15) have been derived assuming that the beam is subjected to pure bending. These equations are exact for pure

6-2.1

FORTRAN Computer Program for Unsymmetrical Bending

```
      PROGRAM UNSYB(INPUT,OUTPUT,UNIN,UNOUT,TAPE5 = UNIN,TAPE6 = UNOUT)
    C READ NUMBER OF CASES
      READ (5,*) NN
      DO 23 J = 1,NN
      WRITE (6,24)
24    FORMAT ("   I(X)   I(Y)     I(XY)     I(MAX)   I(MIN)      M
     APHI   THETA   ALPHA")
    C INPUT N, MOMENTS OF INERTIA, MOMENT AND ANGLE OF PLANE OF LOADS
      READ (5,*) N,EIX,EIXY,EM,PHI
    C COMPUTE PRINCIPAL AXES ANGLE THETA
      B = ABS(EIX - EIY)
      IF (B.GT..001) GO TO 1
      B = .000001
1     THETA = .5*ATAN(-2.*EIXY/B)
    C COMPUTE NEUTRAL AXES ANGLE ALPHA
      EIMAX = .5*(EIX + EIY) + SQRT(.25*(EIX - EIY)**2 + EIXY**2)
      EIMIN = .5*(EIX + EIY) - SQRT(.25*(EIX - EIY)**2 + EIXY**2)
      EMX = EM*SIN(PHI)
      TANAL = (EIXY - EIX/TAN(PHI))/(EIY - EIXY/TAN(PHI))
      ALPHA = ATAN(TANAL)
      A = EMX/(EIX - EIXY*TANAL)
      JJ = J
      WRITE (6,30)JJ
      WRITE (6,26) EIX,EIY,EIXY,EIMAX,EIMIN,EM,PHI,THETA,ALPHA
30    FORMAT("CASE ",JJ)
26    FORMAT (6E9,3,3F7,3,/,/)
      WRITE(6,27)
    C COMPUTE SIGMA ZZ AT N COORDINATE PAIRS (X,Y)
```

```
27    FORMAT ("      X      Y      SIGZZ")
      DO 28 I = 1,N
      READ (5,*) X,Y
      SIGZZ = A*(Y - X* TANAL)
      WRITE (6,32) X,Y,SIGZZ
32    FORMAT (3F10.2)
28    CONTINUE
23    CONTINUE
      STOP
      END
```

INPUT

```
2
1,.5625E + 09,.3906E + 09,.0,.3142E + 08,1.7453
- 125.,150.
2,.1244E + 09,.1244E + 09,.7076E + 08, - .4000E + 08,1.5708
- 59.24, - 215.76
- 84.24,84.24
```

OUTPUT

```
       I(X)         I(Y)        I(XY)      I(MAX)      I(MIN)         M          PHI    THETA   ALPHA
   .563E + 09   .391E + 09     0.        .563E + 09  .391E + 09   .314E + 08    1.745   0.000   .249
N      X            Y           SIGZZ
1   - 125.00      150.00        10.00
       I(X)         I(Y)        I(XY)      I(MAX)      I(MIN)         M          PHI    THETA   ALPHA
   .124E + 09   .124E + 09   .708E + 08  .195E + 09  .536E + 08  - .400E + 08   1.571   - .785  .517
N      X            Y           SIGZZ
1   - 59.24     - 215.76        86.54
2   - 84.24       84.24       - 62.82
```

bending. Although they are not exact for beams subjected to transverse shear loads, often the equations are assumed to be valid for such beams. The error in this assumption is usually small, particularly if the beam has a length of at least five times its maximum cross-sectional dimension.

In the derivation of Eqs. (6-2.14) and (6-2.15), the (x, y)-axes are any convenient set of orthogonal axes that have an origin at the centroid of the cross-sectional area. The equations are valid if (x, y) are principal axes; in this case $I_{xy} = 0$. If the axes are principal axes and $\phi = \pi/2$, Eq. (6-2.14) indicates that $\alpha = 0$ and Eq. (6-2.15) reduces to Eq. (6-1.1).

For convenience in deriving Eqs. (6-2.14) and (6-2.15), the origin for the x, y, z coordinate axes was chosen (see Fig. 6-2.1b) at the end of the free body diagram opposite from the cut section with the positive z-axis toward the cut section. The equations are equally valid if the origin is taken at the cut section with the positive z-axis toward the opposite end of the free body diagram. If ϕ_2 is the magnitude of ϕ for the second choice of axes and ϕ_1 is the magnitude of ϕ for the first choice of axes, then $\phi_2 = \pi - \phi_1$.

A FORTRAN digital computer program for the solution of the nonsymmetrical bending of beams (Eq. 6-2.15) is listed in Table 6-2.1. It can easily be run on a microcomputer.

EXAMPLE 6-2.1

Channel Section Beam

The cantilever beam in Fig. E6-2.1a has a channel section as shown in Fig. E6-2.1b. The concentrated load $P = 12.0$ kN lies in the plane (the plane of the loads) making an angle $\theta = \pi/3$ rad with the x-axis. Load P lies in the plane of the cross section of the free end of the beam and passes through shear center C; in Chapter 7 we find that the shear center lies on the y-axis as shown. Locate points of maximum tensile and compressive stresses in the beam and determine their magnitudes

SOLUTION

Several properties of the cross-sectional area are needed (see the Appendix).

$$A = 10{,}000 \text{ mm}^2 \qquad I_x = 39.69 \times 10^6 \text{ mm}^4$$

$$y_0 = 82.0 \text{ mm} \qquad I_y = 30.73 \times 10^6 \text{ mm}^4$$

$$I_{xy} = 0$$

The orientation of the neutral axis for the beam is given by Eq. (6-2.14). Thus,

$$\tan\alpha = -\frac{I_x}{I_y}\cot\phi = -\frac{39{,}690{,}000}{30{,}730{,}000}(0.5774) = -0.7457$$

$$\alpha = -0.6407 \text{ rad}$$

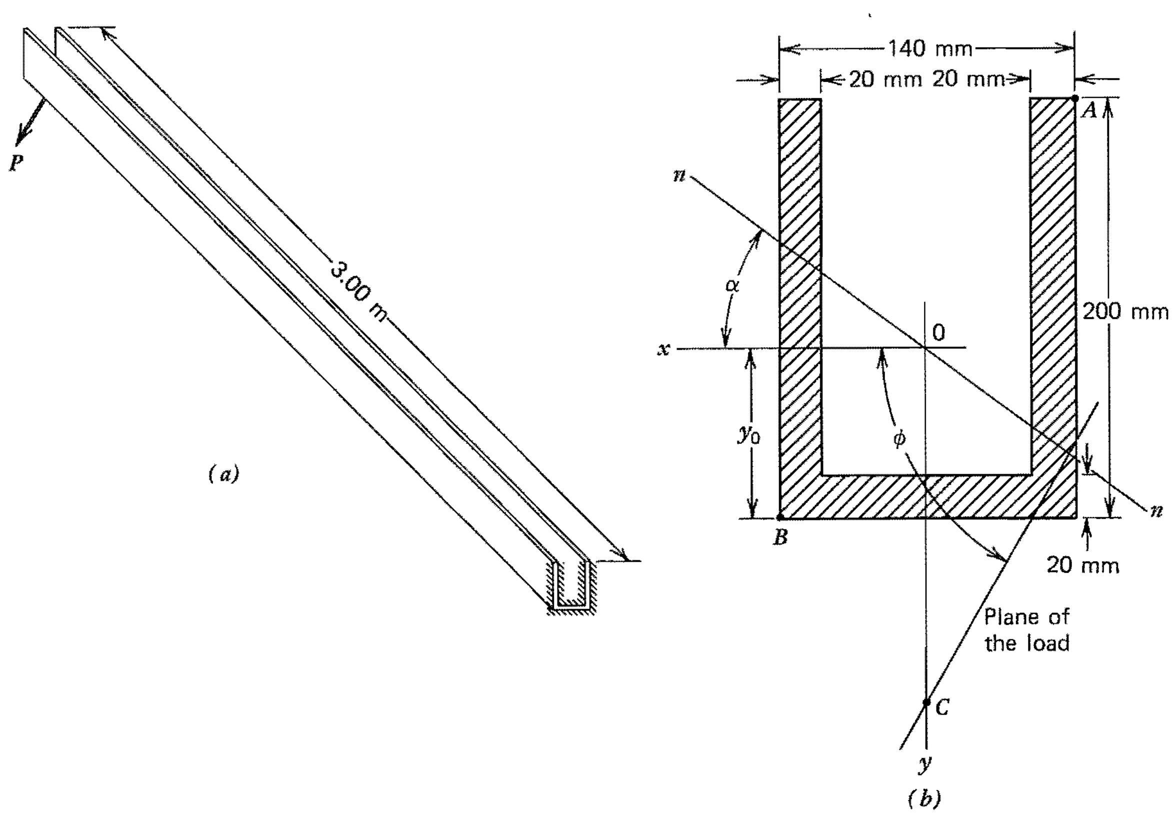


Fig. E6-2.1

The negative sign indicates that the neutral axis *n-n*, which passes through the centroid ($x = y = 0$), is located clockwise 0.6407 rad from the x-axis (Fig. E6-2.1*b*). The maximum tensile stress occurs at point A while the maximum compressive stress occurs at point B. These stresses are given by Eq. (6-2.15) after M_x has been determined. From Fig. E6-2.1*a*

$$M = -3.00P = -36.0 \text{ kN} \cdot \text{m}$$
$$M_x = M \sin\phi = -31.18 \text{ kN} \cdot \text{m}$$
$$\sigma_A = \frac{M_x(y_A - x_A \tan\alpha)}{I_x} = \frac{-31{,}180{,}000[-118 - (-70)(-0.7457)]}{39{,}640{,}000}$$
$$= 133.7 \text{ MPa}$$
$$\sigma_B = \frac{-31{,}180{,}000[82 - 70(-0.7457)]}{39{,}640{,}000} = -105.6 \text{ MPa}$$

EXAMPLE 6-2.2

Angle Beam

Plates are welded together to form the 120 mm by 80 mm by 10 mm angle-section beam shown in Fig. E6-2.2*a*. The beam is subjected to a concentrated load $P = 4.00$ kN as shown. The load P lies in the plane (the plane of loads) making an angle $\phi = 2\pi/3$ rad with the x-axis. Load P passes through shear center C; in Chapter 7 we find that the shear center is located at the intersection of the two legs of the angle section.

Determine the maximum tensile and compressive bending stresses at the section of the beam where the load is applied. (a) Solve the problem using the load-stress relations derived for unsymmetrical bending. (b) Solve the problem using Eq. (6-1.3).

SOLUTION

(a) Several properties of the cross-sectional area are needed (see the Appendix).

$$A = 1900 \text{ mm}^2 \qquad I_x = 2.783 \times 10^6 \text{ mm}^4$$
$$x_0 = 19.74 \text{ mm} \qquad I_y = 1.003 \times 10^6 \text{ mm}^4$$
$$y_0 = 39.74 \text{ mm} \qquad I_{xy} = -0.973 \times 10^6 \text{ mm}^4$$

The orientation of the neutral axis for the beam is given by Eq. (6-2.14). Thus,

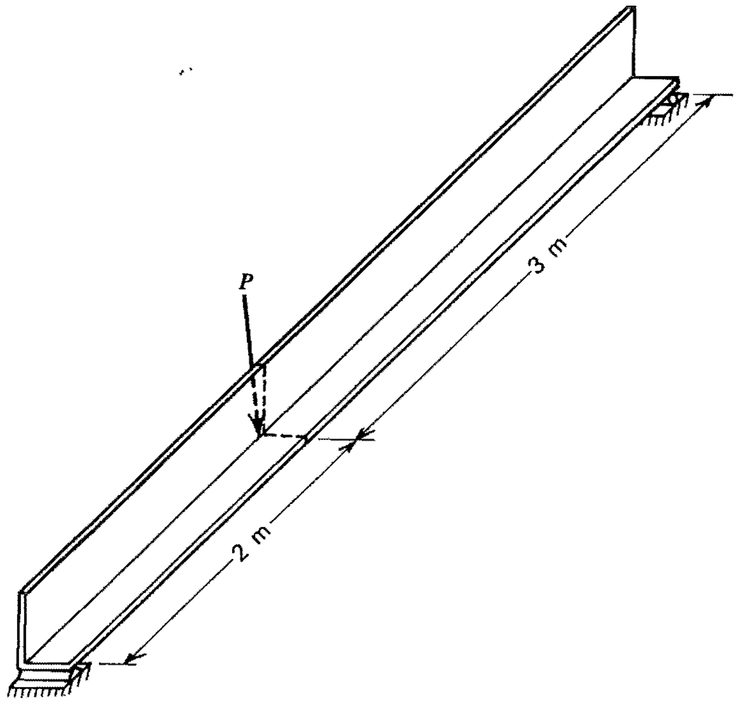
$$\tan\alpha = \frac{I_{xy} - I_x \cot\phi}{I_y - I_{xy}\cot\phi}$$
$$= \frac{-0.973 \times 10^6 - 2.783 \times 10^6(-0.5774)}{1.003 \times 10^6 - (-0.973 \times 10^6)(-0.5774)} = 1.4368$$
$$\alpha = 0.9628 \text{ rad}$$

The positive sign indicates that the neutral axis n-n, which passes through the centroid ($x = y = 0$), is located counterclockwise 0.9628 rad from the x-axis (Fig. E6-2.2b). The maximum tensile stress occurs at point A while the maximum compressive stress occurs at point B. These stresses are given by Eq. (6-2.15) after M_x has been determined. From Fig. E6-2.2a

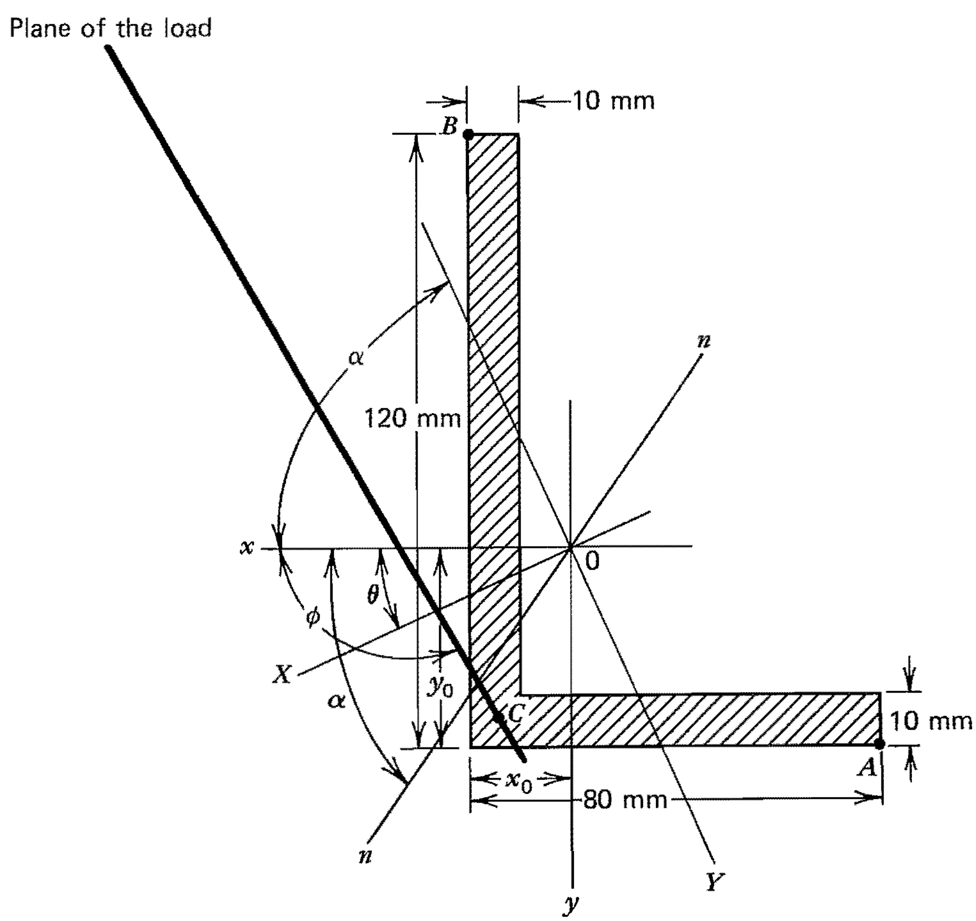
$$M = 1.2P = 4.80 \text{ kN} \cdot \text{m}$$
$$M_x = M \sin\phi = 4.80 \times 10^3\ (0.8660) = 4.157 \text{ kN} \cdot \text{m}$$
$$\sigma_A = \frac{M_x[y_A - x_A \tan\alpha]}{I_x - I_{xy}\tan\alpha}$$
$$= \frac{4.157 \times 10^6[39.74 - (-60.26)(1.4368)]}{2.783 \times 10^6 - (-0.973 \times 10^6)(1.4368)} = 125.6 \text{ MPa}$$
$$\sigma_B = \frac{4.157 \times 10^6[-80.26 - 19.74(1.4368)]}{2.783 \times 10^6 - (-0.973 \times 10^6)(-0.5801)} = -108.0 \text{ MPa}$$

(b) To solve the problem using Eq. (6-1.3), it is necessary that the principal axes for the cross section be determined. The two values of the angle θ between the x-axis and the principal axes are given by Eq. (A-3.4). Thus, we obtain

$$\tan 2\theta = -\frac{2I_{xy}}{I_x - I_y} = -\frac{2(-0.973 \times 10^6)}{2.783 \times 10^6 - 1.003 \times 10^6} = 1.0933$$
$$\theta = 0.4150 \text{ rad}$$


P
3 m
2 m

(a)


Plane of the load
10 mm
B
n
α
120 mm
x
0
θ
φ
X
y_0
α
C
10 mm
A
x_0
80 mm
n
Y
y

(b)

Fig. E6-2.2

The principal X- and Y-axes are shown in Fig. E6-2.2b. Thus (see Eq. A-3.2, the Appendix)

$$I_X = I_x \cos^2\theta + I_y \sin^2\theta - 2I_{xy}\sin\theta\cos\theta = 3.212\times 10^6 \text{ mm}^4$$

$$I_Y = I_x + I_y - I_X = 0.574\times 10^6 \text{ mm}^4$$

Note that now angle ϕ is measured from the X-axis and not from the x-axis as for part (a). Hence

$$\phi = 2\pi/3 - \theta = 1.6794 \text{ rad}$$

Angle α, determines the orientation of the neutral axis, is now measured from the X-axis and is given by Eq. (6-2.14). Hence, we find

$$\tan\alpha = -\frac{I_X \cot\phi}{I_Y} = -\frac{3.212\times 10^6(-0.1090)}{0.574\times 10^6} = 0.6099$$

$$\alpha = 0.5477 \text{ rad}$$

which gives the same orientation for the neutral axis as for part (a).

To use Eq. (6-1.3) relative to axes (X, Y), the X- and Y-coordinates of points A and B are needed. They are

$$X_A = x_A\cos\theta + y_A\sin\theta = -60.26(0.9151) + 39.74(0.4032) = -39.12 \text{ mm}$$

$$Y_A = y_A\cos\theta - x_A\sin\theta = 39.74(0.9151) - (-60.26)(0.4032) = 60.66 \text{ mm}$$

and

$$X_B = 19.74(0.9151) - 80.26(0.4032) = -14.30 \text{ mm}$$

$$Y_B = -80.26(0.9151) - 19.74(0.4032) = -81.41 \text{ mm}$$

The moment components are

$$M_X = M\sin\phi = 4.80\times 10^3(0.9941) = 4.772 \text{ kN}\cdot\text{m}$$

$$M_Y = -M\cos\phi = -4.80\times 10^3(-0.1084) = 520 \text{ N}\cdot\text{m}$$

The stresses at A and B are calculated using Eq. (6-1.3). Thus,

$$\sigma_A = \frac{M_X Y_A}{I_X} - \frac{M_Y X_A}{I_Y}$$

$$= \frac{4.772\times 10^6(60.66)}{3.212\times 10^6} - \frac{0.520\times 10^6(-39.12)}{0.574\times 10^6} = 125.6 \text{ MPa}$$

$$\sigma_B = \frac{M_X Y_B}{I_X} - \frac{M_Y X_B}{I_Y}$$

$$= \frac{4.772\times 10^6(-81.41)}{3.212\times 10^6} - \frac{0.520\times 10^6(-14.30)}{0.574\times 10^6} = -108.0 \text{ MPa}$$

These values for σ_A and σ_B agree with the values calculated in part (a). However, the computational work is greater in part (b) than in part (a).

PROBLEM SET 6-2

1. A timber beam 250 mm wide by 300 mm deep by 4.2 m long is used as a simple beam on a span of 4 m. It is subjected to a concentrated load P at the midsection of the span. The plane of the loads makes an angle $\phi = 5\pi/9$ rad with the horizontal x-axis. The beam is made of yellow pine with a yield stress $Y = 25.0$ MPa. If the beam has been designed with a factor of safety $SF = 2.50$ against initiation of yielding, determine the magnitude of P and the orientation of the neutral axis.

2. The plane of the loads for the rectangular section beam in Fig. P6-2.2 coincides with a diagonal of the rectangle. Show that the neutral axis for the beam cross section coincides with the other diagonal.

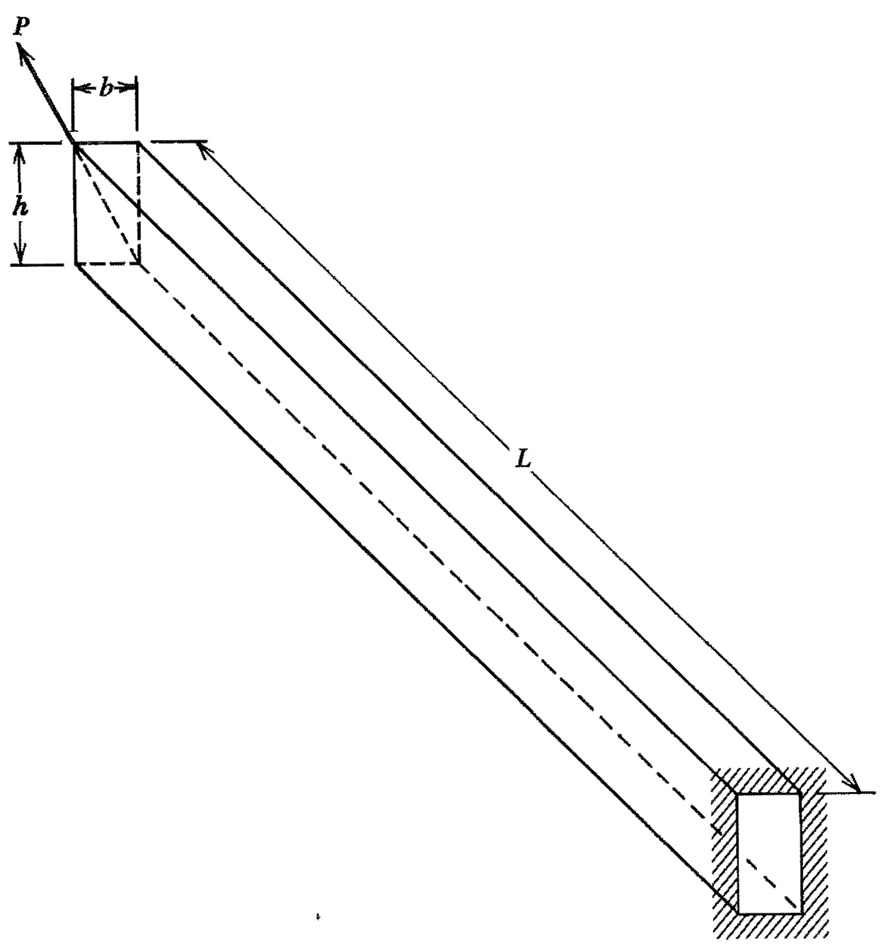


Fig. P6-2.2

3. In Fig. P6-2.3 let $b = 300$ mm, $h = 300$ mm, $t = 25.0$ mm, $L = 2.50$ m, and $P = 16.0$ kN. Calculate the maximum tensile and compressive stresses in the beam, and determine the orientation of the neutral axis.

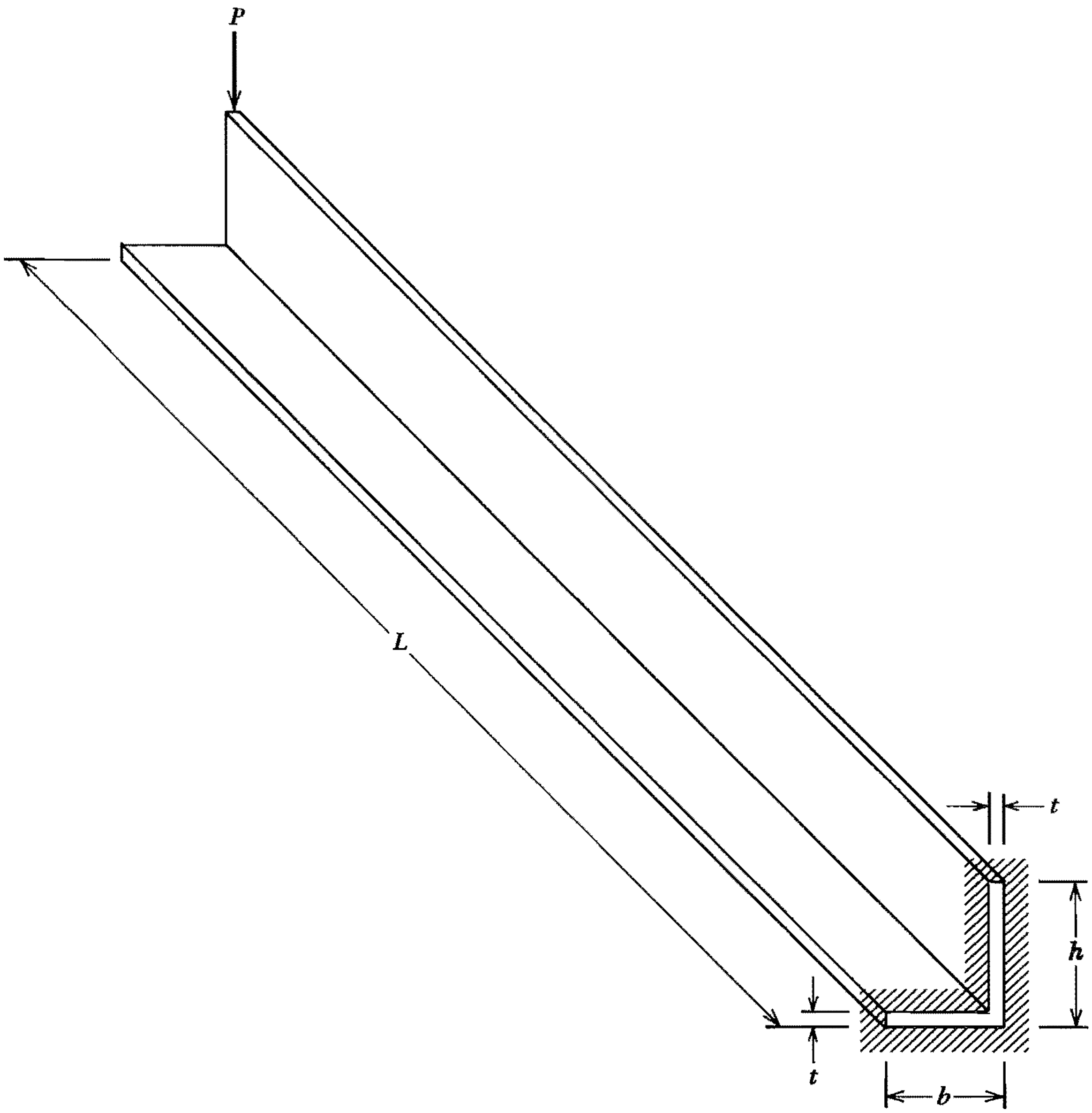


Fig. P6-2.3

4. In Fig. P6-2.3 let $b = 200$ mm, $h = 300$ mm, $t = 25.0$ mm, $L = 2.50$ m, and $P = 16.0$ kN. Calculate the maximum tensile and compressive stresses in the beam and determine the orientation of the neutral axis.

Ans. $\sigma_{zz(\text{ten})} = 98.6$ MPa, $\sigma_{zz(\text{com})} = -81.9$ MPa

5. In Fig. P6-2.5 let $b = 150$ mm, $t = 50.0$ mm, $h = 150$ mm, and $L = 2.00$ m. The beam is made of a steel that has a yield point stress $Y = 240$ MPa. Using a factor of safety of $SF = 2.00$, determine the magnitude of P if $\phi = 2\pi/9$ rad from the horizontal x-axis.

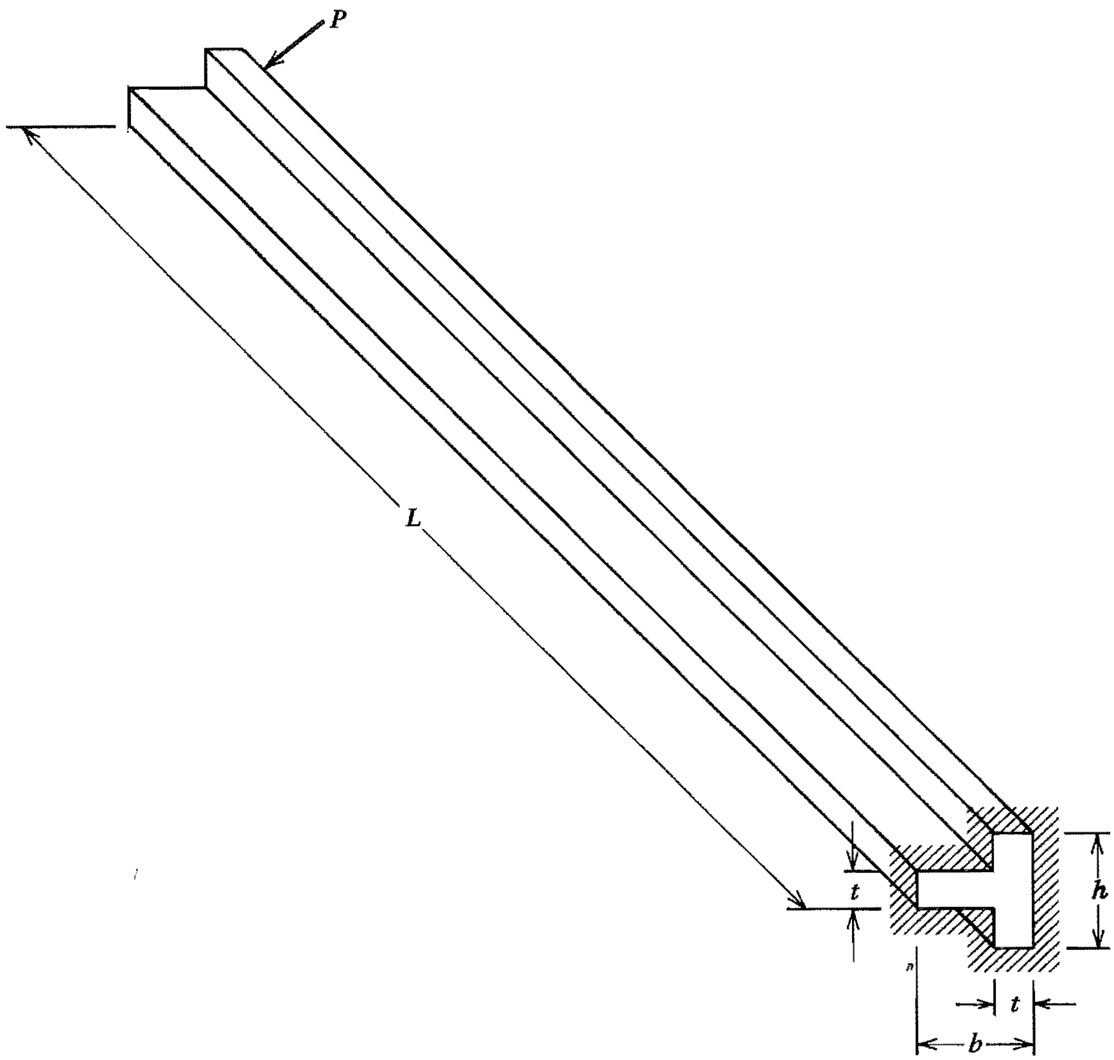


Fig. P6-2.5

6. A simple beam is subjected to a concentrated load $P = 4.00$ kN at the midlength of a span of 2.00 m. The beam cross section is formed by nailing two 50.0 mm by 150 mm boards together as indicated in Fig. P6-2.6. The plane of the loads passes through the centroid of the two boards as indicated. Determine the maximum flexure stress in the beam and the orientation of the neutral axis.

   Ans. $\sigma_{zz(\max)} = 4.17$ MPa, $\alpha = 1.3522$ rad

7. Solve Problem 6 if $\phi = 1.900$ rad.

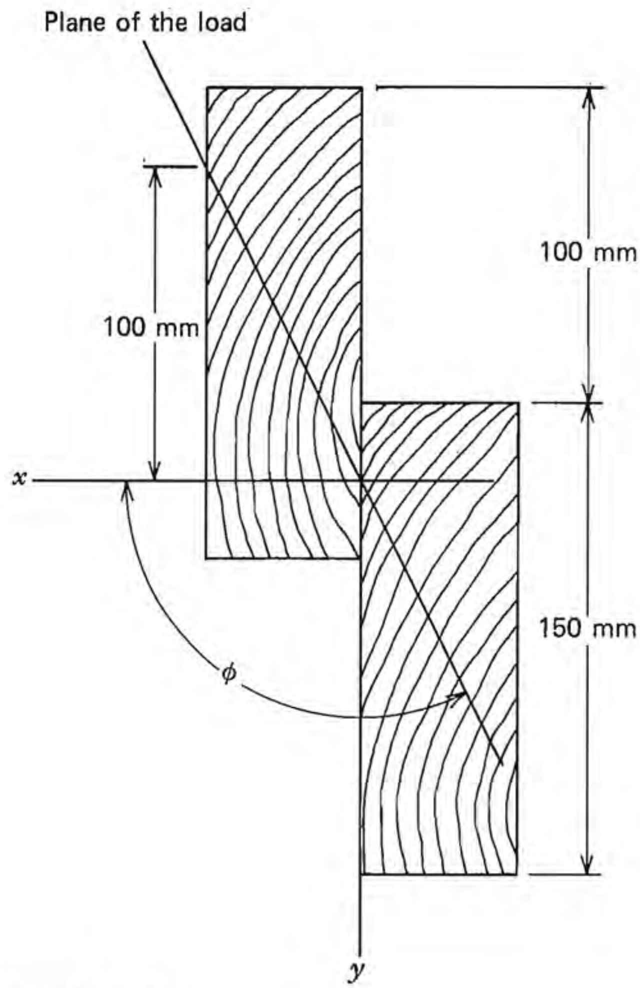


Fig. P6-2.6

8. A C-180 × 14.6 rolled steel channel ($I_x = 8.87 \times 10^6$ mm^4, depth = 178 mm, width = 53 mm, $x_B = 13.7$ mm) is used as a simply supported beam as, for example, a purlin in a roof (Fig. P6-2.8). If the slope of the roof is 1/2 and the span of the purlin is 4 m, determine the maximum tensile and compressive stresses in the beam caused by a uniformly distributed vertical load of 1.00 kN/m.

Ans. $\sigma_{zz(\text{ten})} = 48.4$ MPa, $\sigma_{zz(\text{com})} = -105.2$ MPa

9. Two L-89 × 64 × 7.9 rolled steel angles ($I_{x_1} = 391 \times 10^3$ mm^4, $I_{y_1} = 912 \times 10^3$ mm^4, $I_{x_1y_1} = 349 \times 10^3$ mm^4, and $A = 1148$ mm^2) are welded to a 200 mm by 10 mm steel plate to form a composite z-bar (Fig. P6-2.9). The z-bar is a simply supported beam used as a purlin in a roof of slope $\frac{1}{2}$. The beam has a span of 4.00 m. The yield stress of the steel in the plate and angles is $Y = 300$ MPa. The beam has been designed using a factor of safety of $SF = 2.50$ against initiation of yielding. If the plane of the loads is vertical, determine the

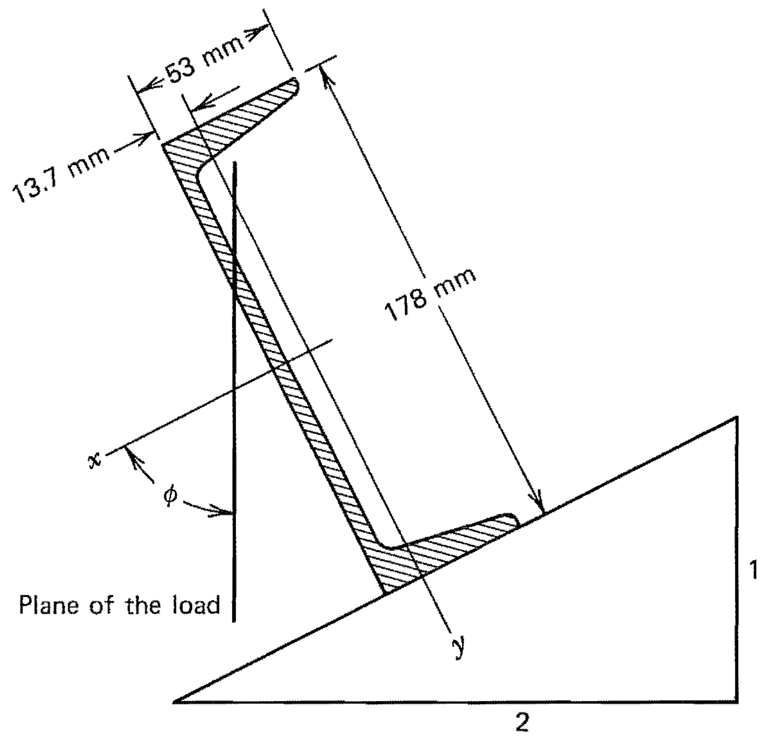


Fig. P6-2.8

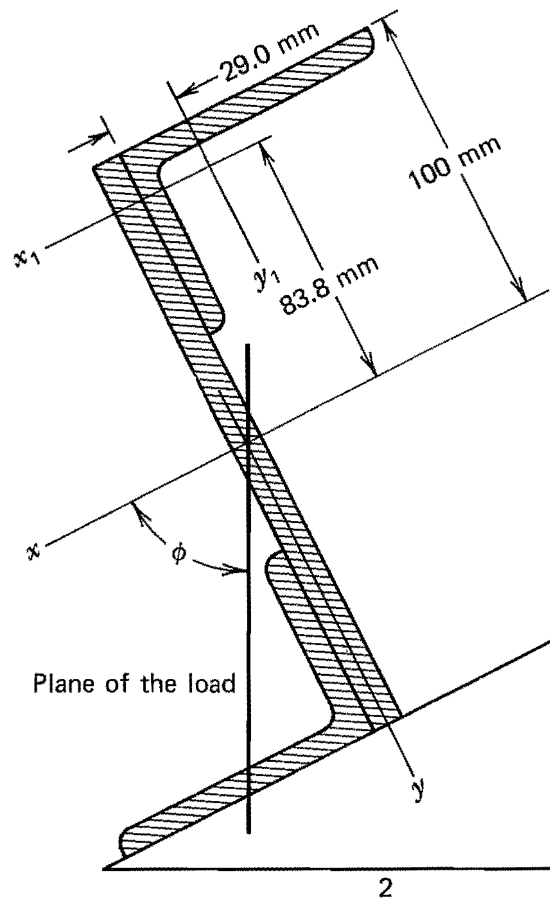


Fig. P6-2.9

magnitude of the maximum distributed load that can be applied to the beam.

10. A steel z-bar is used as a centilever beam having a length of 2.00 m. Looking from the free end toward the fixed end of the beam, the cross section has the orientation and dimensions shown in Fig. P6-2.10. A concentration load $P = 14.0$ kN acts at the free end of the beam at an angle $\phi = 1.25$ rad. Determine the maximum flexure stress in the beam.

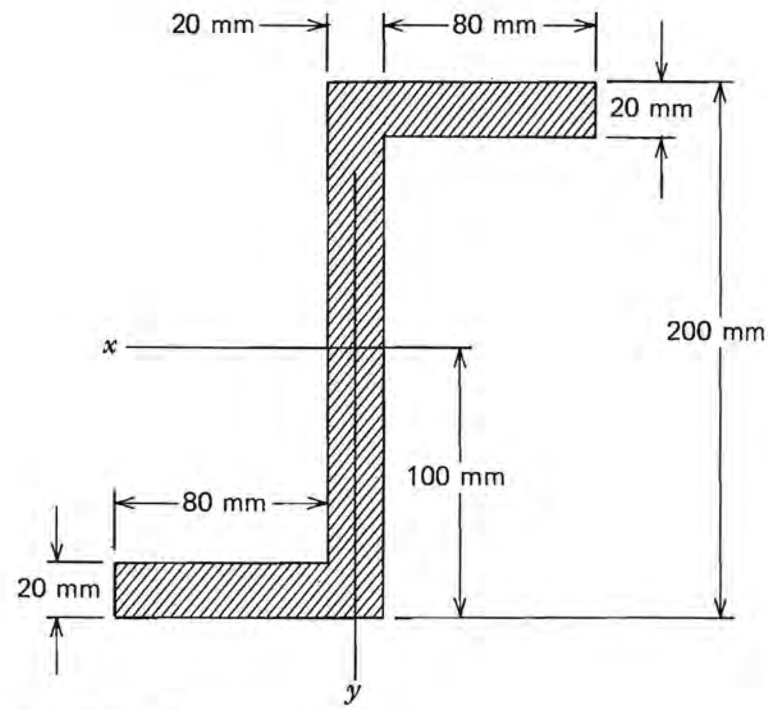


Fig. P6-2.10

Ans. $I_x = 39.36 \times 10^6$ mm^4, $I_y = 9.84 \times 10^6$ mm^4, $I_{xy} = 14.40 \times 10^6$ mm^4, $\alpha =$ 0.2557 rad, $\sigma_{zz(\max)} = 76.6$ MPa.

11. An extruded bar of aluminum alloy has the cross section shown in Fig. P6-2.11. A 1.00 m length of this bar is used as a cantilever beam. A concentrated load $P = 1.25$ kN is applied at the free end and makes an angle of $\phi = 5\pi/9$ rad with the x-axis. The view in Fig. P6-2.11 is from the free end toward the fixed end of the beam. Determine the maximum tensile and compressive stresses in the beam.

12. An extruded bar of aluminum alloy has the cross section shown in Fig. P6-2.12. A 2.10 m length of this bar is used as a simple beam on a span of 2.00 m. A concentrated load $P = 5.00$ kN is applied at

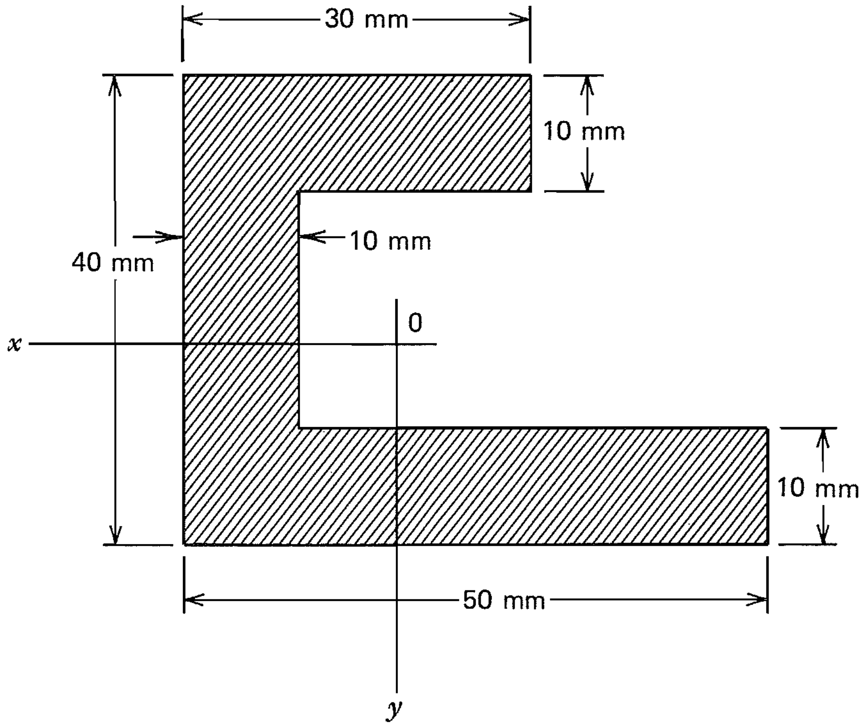


Fig. P6-2.11

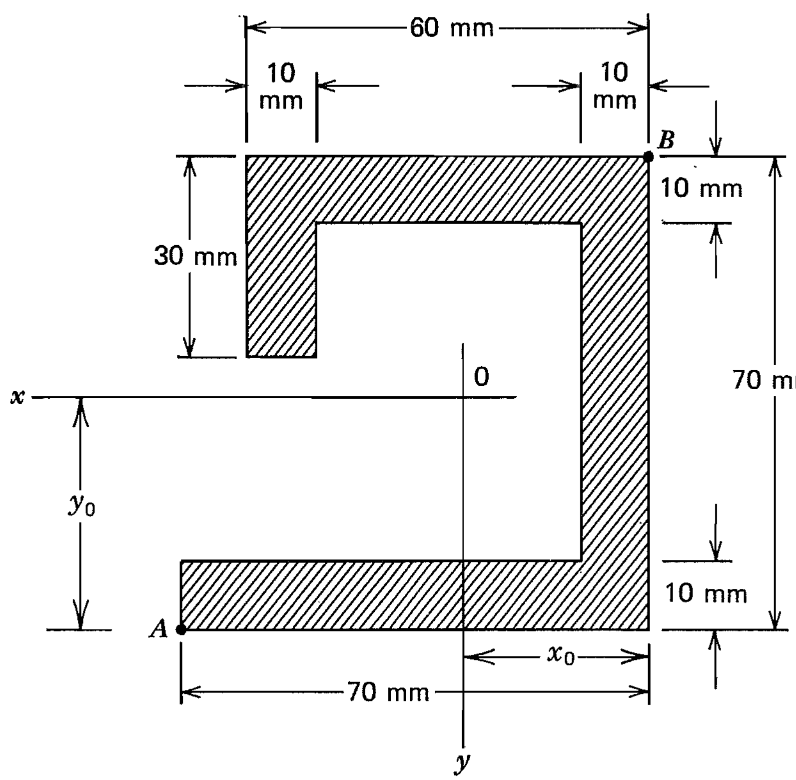


Fig. P6-2.12

midlength of the span and makes an angle of $\phi = 1.40$ rad with the x-axis. Determine the maximum tensile and compressive stresses in the beam.

Ans. $x_0 = 28.0$ mm, $y_0 = 35.0$ mm, $I_x = 1.330 \times 10^6$ mm^4,
$I_y = 917 \times 10^3$ mm^4, $I_{xy} = 30.0 \times 10^3$ mm^4,
$\alpha = -0.2153$ rad, $\sigma_A = 91.5$ MPa, $\sigma_B = -75.8$ MPa

13. A cantilever beam has a right triangular cross section and is loaded by a concentrated load P at the free end (Fig. P6-2.13). Solve for the stresses at points A and C at the fixed end if $P = 4.00$ kN, $h = 120$ mm, $b = 75.0$ mm, and $L = 1.25$ m.

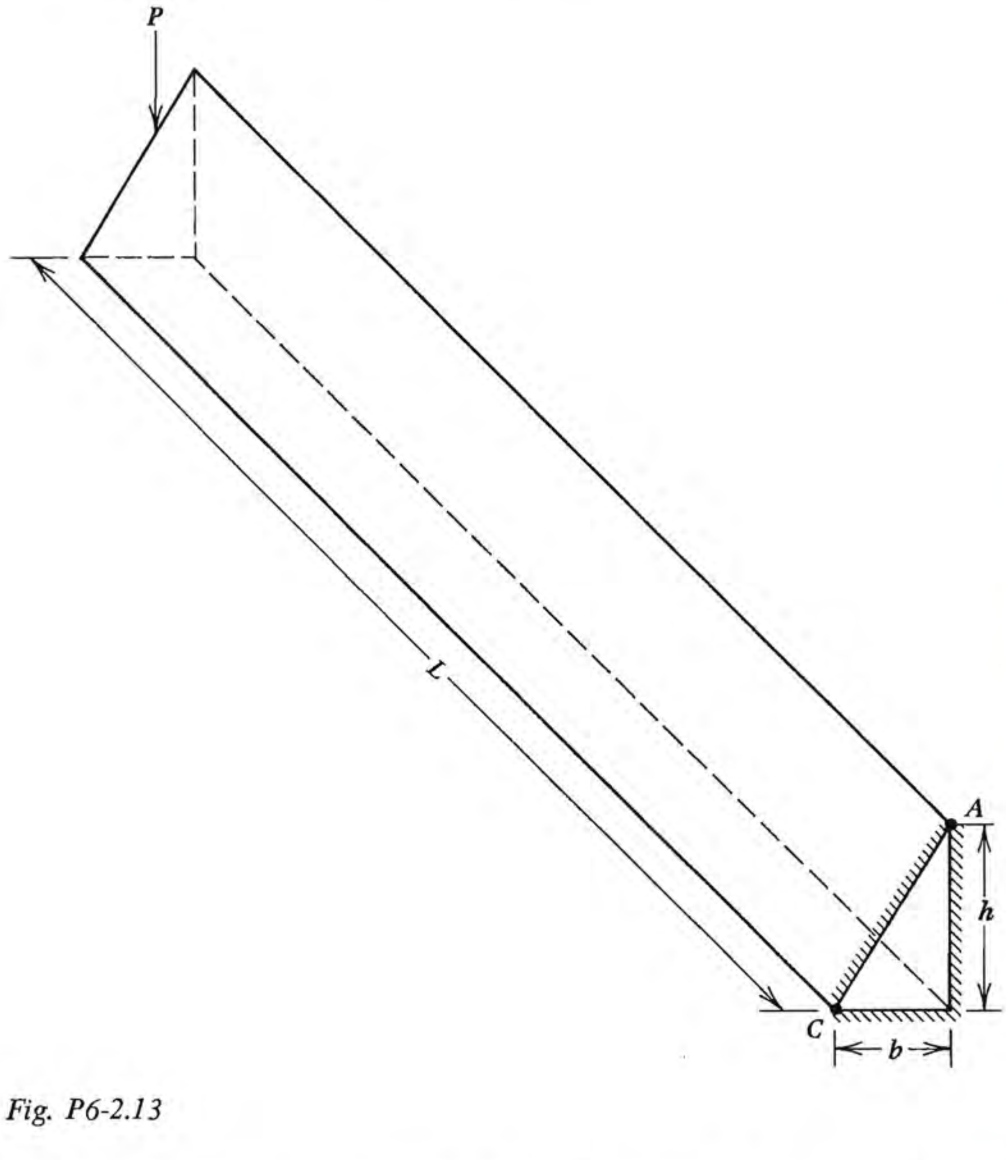


Fig. P6-2.13

14. A girder that supports a brick wall is built up of an S-310 × 47.3 I-beam ($A_1 = 6032$ mm^2, $I_{x_1} = 90.7 \times 10^6$ mm^4, $I_{y_1} = 3.90 \times 10^6$

mm^4), a C-310 × 30.8 channel ($A_2 = 3929\ \text{mm}^2$, $I_{x_2} = 53.7 \times 10^6\ \text{mm}^4$, $I_{y_2} = 1.61 \times 10^6\ \text{mm}^4$), and a cover plate 300 mm by 10 mm riveted together (Fig. P6-2.14). The girder is 6.00 m long and is simply supported at its ends. The load is uniformly distributed such that $w = 20.0$ kN/m. Determine the orientation of the neutral axis and the maximum tensile and compressive stresses.

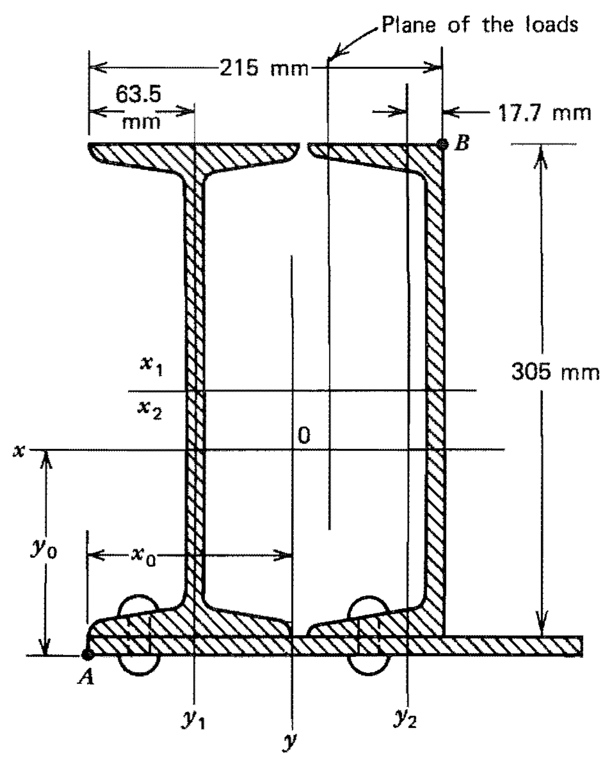


Fig. P6-2.14

Ans. $\alpha = -0.1653$ rad, $\sigma_A = 66.3$ MPa, $\sigma_B = -92.3$ MPa

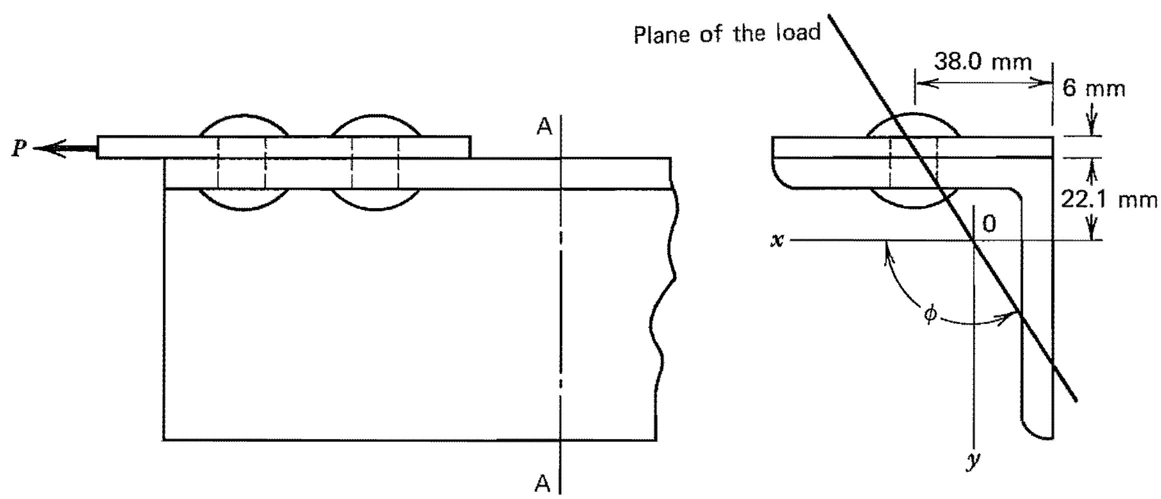


Fig. P6-2.15

15. A load $P = 50$ kN is applied to an L-76 × 76 × 7.9 rolled steel angle ($I_x = I_y = 570 \times 10^3$ mm^4, $I_{xy} = -332.5 \times 10^3$ mm^4, $A = 1148$ mm^2) by means of a 76 mm by 6 mm plate riveted to the angle (Fig. P6-2.15). The action line of load P coincides with the centroidal axis of the plate. Determine the maximum stress at a section, such as AA, of the angle. *Hint:* Resolve the load P into a load (equal to P) at the centroid of the angle and a bending couple.

6-3 DEFLECTIONS OF STRAIGHT BEAMS SUBJECTED TO UNSYMMETRICAL BENDING

If every transverse shear load applied to a straight beam lies in one plane and if every couple applied to the beam lies in a plane parallel to the plane of the transverse shear loads, the neutral axis for every cross section of the beam will have the same orientation as long as the beam material remains linearly elastic. The deflections of the beam will be in a direction perpendicular to the neutral axis. It is convenient to determine the component of the deflection parallel to an axis, say the y-axis. The total deflection is easily determined once one component has been determined.

Consider the intersection of the (y, z)-plane with the beam in Fig. 6-2.1. A side view of this section of the deformed beam is shown in Fig. 6-3.1. In the deformed beam, the two straight lines F^*G^* and H^*J^* represent the intersection of the (y, z)-plane with two planes perpendicular to the axis of the beam, a distance Δz apart at the neutral surface. Before deformation, the lines FG and HJ are parallel and distance Δz apart. Since plane sections remain plane, the extensions of F^*G^* and H^*J^* meet at the center of curvature $0'$. The distance from $0'$ to the neutral surface is the radius of curvature R_y of the beam in the (y, z)-plane. Since the center of curvature lies on the negative side of the y-axis, R_y is negative. We assume that the deflections are small so that $1/R_y \cong d^2v/dz^2$, where v is the y-component of displacement. Under deformation of the beam, a fiber at distance y below the neutral surface elongates an amount $e_{zz} = (\Delta z)\epsilon_{zz}$. Initially the length of the fiber is Δz. By geometry of similar triangles,

$$-\frac{\Delta z}{R_y} = \frac{(\Delta z)\epsilon_{zz}}{y}$$

Dividing by Δz, we obtain

$$-\frac{1}{R_y} = \frac{\epsilon_{zz}}{y} \qquad \frac{1}{R_y} \cong \frac{d^2v}{dz^2} \tag{6-3.1}$$

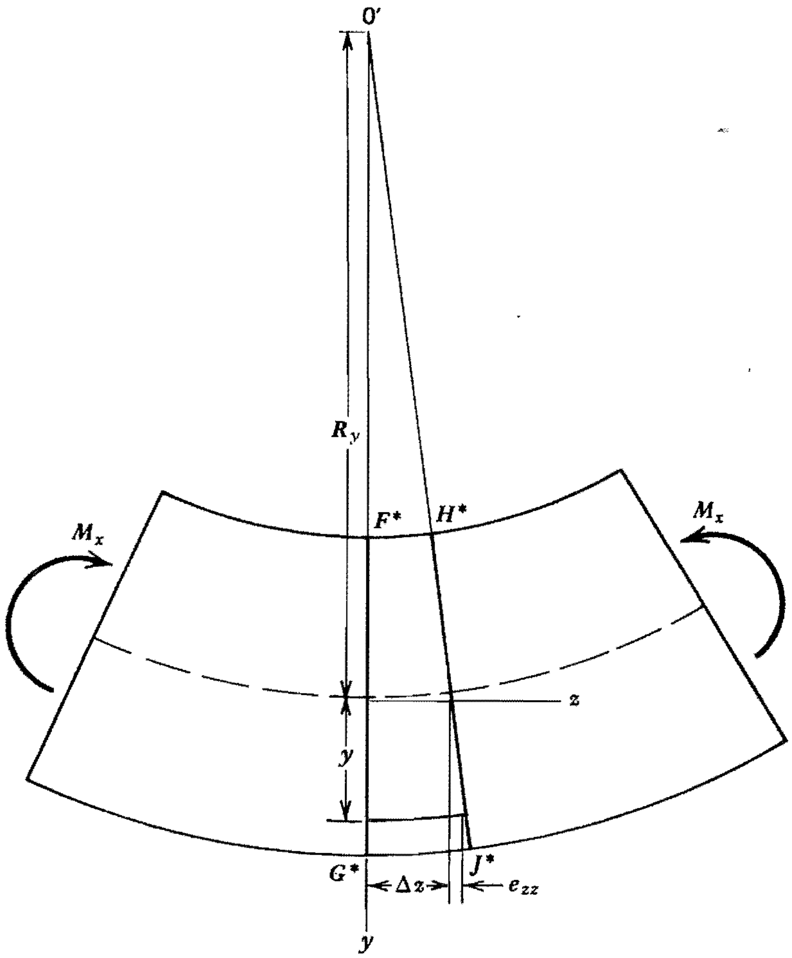


Fig. 6-3.1/Deflection of an unsymmetrically loaded beam.

For linearly elastic behavior, Eqs. (6-2.15) and (6-2.9), with $x = 0$, and Eq. (6-2.4) yield

$$\frac{\epsilon_{zz}}{y} = \frac{M_x}{E(I_x - I_{xy}\tan\alpha)} = \frac{M_x I_y + M_y I_{xy}}{E\left(I_x I_y - I_{xy}^2\right)}$$

which with Eq. (6-3.1) yields

$$\frac{d^2 v}{dz^2} = -\frac{M_x}{E(I_x - I_{xy}\tan\alpha)} = -\frac{M_x I_y + M_y I_{xy}}{E\left(I_x I_y - I_{xy}^2\right)} \qquad (6\text{-}3.2)$$

Note the similarity of Eq. (6-3.2) to the elastic curve equation for symmetrical bending. The only difference is that the term I has been replaced by $(I_x - I_{xy}\tan\alpha)$. The solution of the differential relation Eq. (6-3.2) gives the y-component of the deflection v at any section of the beam. As is indicated in Fig. 6-3.2, the total deflection of the neutral axis at any section of the beam is perpendicular to the neutral axis. Therefore,

$$u = -v\tan\alpha \qquad (6\text{-}3.3)$$

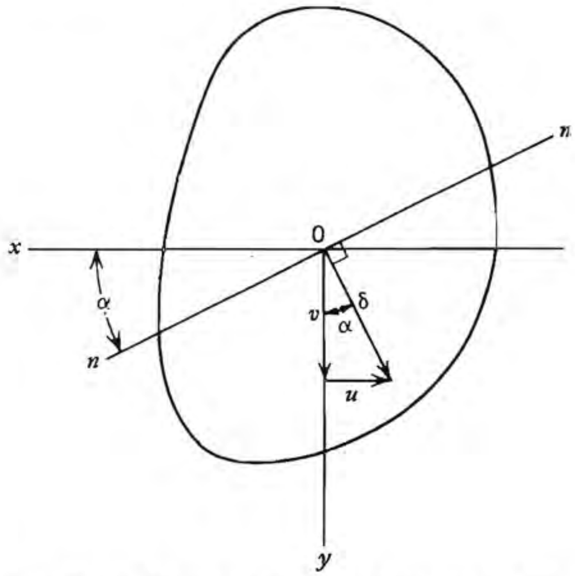


Fig. 6-3.2/Components of deflection of an unsymmetrically loaded beam.

and the total displacement is

$$\delta = \sqrt{u^2 + v^2} = \frac{v}{\cos \alpha} \tag{6-3.4}$$

EXAMPLE 6-3.1

Channel Section Simple Beam

Let the channel section beam in Fig. E6-2.1 be loaded as a simple beam with a concentrated load $P = 35.0$ kN acting at the center of the beam. Determine the maximum tensile and compressive stresses in the beam if $\phi = 5\pi/9$. If the beam is made of an aluminum alloy ($E = 72.0$ GPa), determine the maximum deflection of the beam.

SOLUTION

From Problem E6-2.1

$$\tan \alpha = -\frac{I_x}{I_y} \cot \phi = -\frac{39{,}690{,}000}{30{,}730{,}000} \cot \frac{5\pi}{9} = 0.2277$$

$$\alpha = 0.2239 \text{ rad}$$

$$M = \frac{PL}{4} = \frac{35.0(3.00)}{4} = 26.25 \text{ kN} \cdot \text{m}$$

$$M_x = M \sin \phi = 25.85 \text{ kN} \cdot \text{m}$$

$$\sigma_{\text{tension}} = \frac{25{,}850{,}000[82 - (-70)(0.2277)]}{39{,}690{,}000} = 63.8 \text{ MPa}$$

$$\sigma_{\text{compression}} = \frac{25{,}850{,}000[-118 - 70(0.2277)]}{39{,}690{,}000} = -87.2 \text{ MPa}$$

Since the deflection of the center of a simple beam subjected to a concentrated load in the center is given by the relation $PL^3/48EI$, the y-component of the deflection of the center of the beam is

$$v = \frac{PL^3 \sin\phi}{48EI_x} = \frac{35{,}000(3000)^3 \sin 5\pi/9}{48(72{,}000)(39{,}690{,}000)} = 6.78 \text{ mm}$$

$$u = -v \tan\alpha = -6.78(0.2277) = -1.54 \text{ mm}$$

$$\delta = \sqrt{u^2 + v^2} = 6.95 \text{ mm}$$

EXAMPLE 6-3.2

Cantilever I-Beam

A cantilever beam has a length of 3 m with cross section indicated in Fig. E6-3.2. The beam is constructed by welding two 40 mm by 40 mm steel ($E = 200$ GPa) bars longitudinally to the S-200 × 27.4 steel I-beam ($I_x = 24 \times 10^6$ mm^4 and $I_y = 1.55 \times 10^6$ mm^4). The bars and I-beam have the same yield stress, $Y = 300$ MPa. The beam is subjected to a concentrated load P at the free end at an angle $\phi = \pi/3$ rad with the x-axis. Determine the magnitude of P necessary to initiate yielding in the beam and the resulting deflection of the free end of the beam.

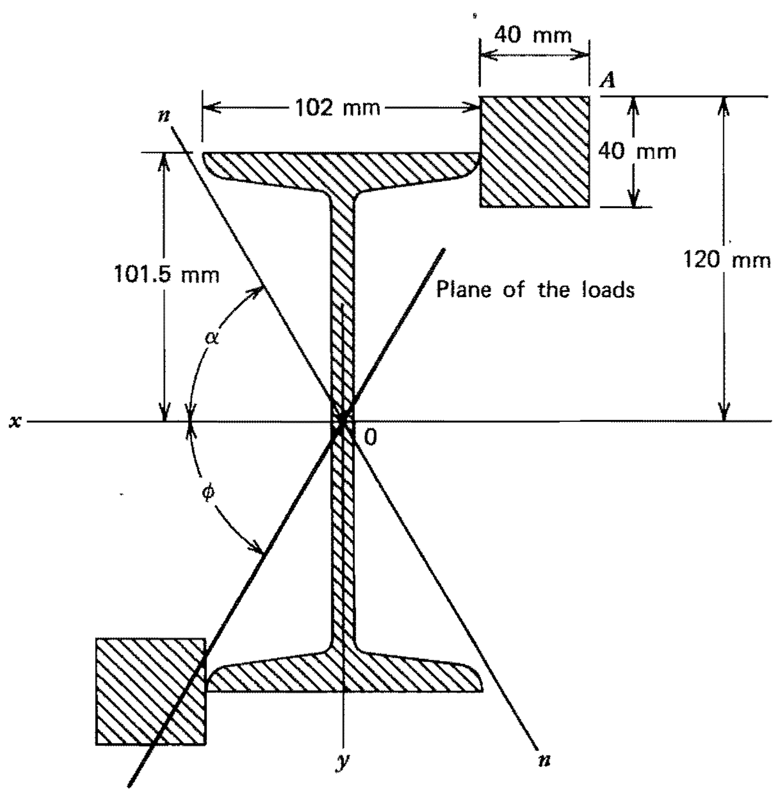


Fig. E6-3.2

SOLUTION

Values of I_x, I_y, and I_{xy} for the composite cross section can be obtained using the procedure outlined in the Appendix.

$$I_x = 56.43 \times 10^6 \text{ mm}^4 \qquad I_y = 18.11 \times 10^6 \text{ mm}^4$$

$$I_{xy} = 22.72 \times 10^6 \text{ mm}^4$$

The orientation of the neutral axis for the beam is given by Eq. (6-2.14). Hence, we find

$$\tan\alpha = \frac{I_{xy} - I_x \cot\phi}{I_y - I_{xy}\cot\phi} = \frac{22.72 \times 10^6 - 56.43 \times 10^6(0.5774)}{18.11 \times 10^6 - 22.72 \times 10^6(0.5774)} = -1.9759$$

$$\alpha = -1.023 \text{ rad}$$

The orientation of the neutral axis $n - n$ is indicated in Fig. E6-3.1. The maximum tensile stress occurs at point A; the magnitude of the stress is obtained using Eq. (6-2.15).

$$M = -3P \text{ N} \cdot \text{m}$$

$$M_x = M\sin\phi = -2.598P \text{ N} \cdot \text{m}$$

$$\sigma_A = Y = \frac{M_x(y_A - x_A \tan\alpha)}{I_x - I_{xy}\tan\alpha}$$

$$P = \frac{Y(I_x - I_{xy}\tan\alpha)}{(-2.598 \times 10^3)(y_A - x_A \tan\alpha)}$$

$$= \frac{300[56.43 \times 10^6 - 22.72 \times 10^6(-1.9759)]}{-2.598 \times 10^3[-120 - (-91)(-1.9759)]} = 39.03 \text{ kN}$$

Since the deflection of the free end of a cantilever beam subjected to symmetrical bending is given by the relation $P_y L^3/3EI$, the y-component of the deflection of the free end of the beam is

$$v = \frac{PL^3 \sin\phi}{3E(I_x - I_{xy}\tan\alpha)}$$

$$= \frac{39.03 \times 10^3 (3 \times 10^3)^3 (0.8660)}{3(200 \times 10^3)[56.43 \times 10^6 - 22.72 \times 10^6(-1.9759)]} = 17.33 \text{ mm}$$

Hence,

$$u = -v\tan\alpha = 34.25 \text{ mm}$$

and the total displacement of the free end of cantilever beam is

$$\phi = \sqrt{u^2 + v^2} = 38.39 \text{ mm}$$

PROBLEM SET 6-3

1. Determine the deflection of the beam in Problem 6-2.1 if $E = 12.0$ GPa for the yellow pine.

2. The beam in Problem 6-2.3 is made of 7075-T6 aluminum alloy for which $E = 71.7$ GPa. Determine the deflection of the free end of the beam.

   Ans. $v = 13.81$ mm, $u = -7.86$ mm, $\delta = 15.89$ mm

3. The beam in Problem 6-2.4 is made of 7075-T6 aluminum alloy for which $E = 71.7$ GPa. Determine the deflection of the free end of the beam.

4. The beam in Problem 6-2.6 is made of yellow pine for which $E = 12.0$ GPa. Determine the deflection at the center of the beam.

   Ans. $v = 0.33$ mm, $u = -1.49$ mm, $\delta = 1.53$ mm

5. Determine the deflection of the center of the beam in Problem 6-2.8. $E = 200$ GPa.

6. If the beam in Problem 6-2.9 is subjected to a distributed load of $w = 6.5$ kN/m, determine the deflection of the beam at the center of the beam. $E = 200$ GPa.

   Ans. $v = 1.58$ mm, $u = 8.25$ mm, $\delta = 8.40$ mm

7. Determine the deflection of the beam in Problem 6-2.10. $E = 200$ GPa.

8. Determine the deflection of the free end of the beam in Problem 6-2.11. $E = 72.0$ GPa.

   Ans. $v = 33.16$ mm, $u = 6.25$ mm, $\delta = 33.74$ mm

9. Determine the deflection of the midspan of the beam in Problem 6-2.12. $E = 72.0$ GPa.

10. Determine the deflection of the free end of the beam in Problem 6-2.13. $E = 200$ GPa.

    Ans. $v = 4.82$ mm, $u = -3.86$ mm, $\delta = 6.18$ mm

6-4 CHANGE IN DIRECTION OF NEUTRAL AXIS AND INCREASE IN STRESS AND DEFLECTION IN ROLLED SECTIONS DUE TO A VERY SMALL INCLINATION OF PLANE OF LOADS TO A PRINCIPAL PLANE

Some commonly rolled sections such as I-beams and channels are designed so that I_x is many times greater than I_y and $I_{xy} = 0$. Equation (6-2.14) indicates that the angle α may be large even though ϕ is nearly equal to $\pi/2$ rad. Thus, the neutral axis of such I-beams and channels is steeply inclined to the horizontal axis (the x-axis) of symmetry when the plane of the loads deviates slightly from the vertical plane of symmetry. As a consequence, the maximum flexure stress and the maximum deflection may be quite large. These rolled sections should not be used as beams unless the lateral deflection is prevented. If lateral deflection of the beam is prevented, unsymmetrical bending cannot occur.

In general, however, I-beams and channels make very poor cantilever beams. The following example illustrates this fact.

EXAMPLE 6-4.1

An Unsuitable Cantilever Beam

An S-610 × 134 I-beam ($I_x = 937 \times 10^6$ mm^4 and $I_y = 18.7 \times 10^6$ mm^4) is subjected to a bending moment M in a plane with angle $\phi = 1.5533$ rad; the plane of the loads is 1^0 ($\pi/180$ rad) clockwise from the (y, z)-plane of symmetry. Determine the neutral axis and the ratio of the maximum tensile stress in the beam to the maximum tensile stress for symmetrical bending.

SOLUTION

The cross section of the I-beam with the plane of the loads is indicated in Fig. E6-4.1. The orientation of the neutral axis for the beam is given by Eq. (6-2.14).

$$\tan\alpha = \frac{-I_x \cot\phi}{I_y} = -\frac{937 \times 10^6(0.01746)}{18.7 \times 10^6} = -0.8749$$

$$\alpha = -0.7188 \text{ rad}$$

The orientation of the neutral axis is indicated in Fig. E6-4.1. If the beam is subjected to a positive bending moment, the maximum tensile stress is

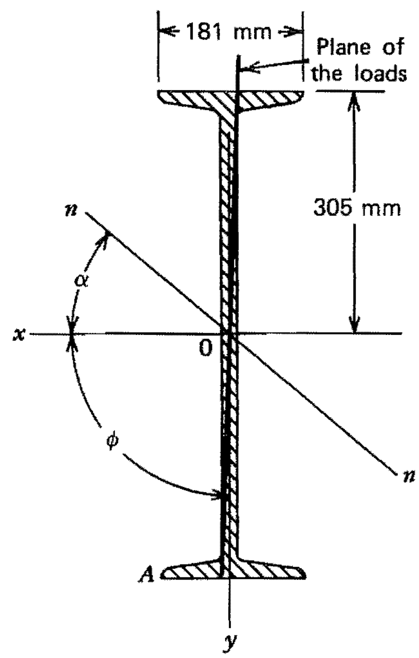


Fig. E6-4.1

located at point A. By Eq. (6-2.10),

$$M_x = M \sin\phi = 0.9998M$$

$$\sigma_A = \frac{0.9998M[305 - 90.5(-0.8749)]}{937 \times 10^6} = 4.099 \times 10^{-7}M \text{ MPa} \quad (1)$$

When the plane of the loads coincide with the y-axis (Fig. E6-4.1), the beam is subjected to symmetrical bending and the maximum bending stress is

$$\sigma_A = \frac{My}{I_x} = \frac{305M}{937 \times 10^6} = 3.255 \times 10^{-7}M \text{ MPa} \quad (2)$$

The ratio of the stress σ_A given by Eq. (1) to that given by Eq. (2) is 1.260. Hence the maximum stress in the I-beam is increased 25.2 percent when the plane of the loads is merely 1^0 from the symmetrical vertical plane.

6-5 FULLY PLASTIC LOAD FOR UNSYMMETRICAL BENDING

A beam of general cross section (Fig. 6-5.1) is subjected to pure bending. The material in the beam has a flat top stress-strain diagram with yield point Y in both tension and compression (Fig. 2-6.2a). At the fully plastic load, the deformations of the beam are unchecked and continue

(until possibly the material begins to strain harden). The fully plastic load is the upper limit for failure loads (Art. 3-3) since the deformations of the beam at the outset of any strain hardening generally exceeds design limits for the deformations.

In contrast to the direct calculation of fully plastic load in symmetrical bending (Art. 3-3), an inverse method is required to determine the fully plastic load for a beam subjected to unsymmetrical bending. Although the plane of the loads is generally specified for a given beam, the orientation and location of the neutral axis, when the fully plastic moment is developed at a given section of the beam, must be determined by trial and error. The analysis is begun by assuming a value for the angle α (Fig. 6-5.1). The neutral axis is inclined to the x-axis by the angle α but does not necessarily pass through the centroid as in the case of linearly elastic conditions. The location of the neutral axis is determined by the condition that it must divide the cross-sectional area into equal parts. This follows from the fact that since the yield point stress is the same for tension and compression, the area A_T that has yielded in tension must be equal to the area A_C that has yielded in compression. In other words, the net resultant tension force on the section must be equal to the net resultant compression force.

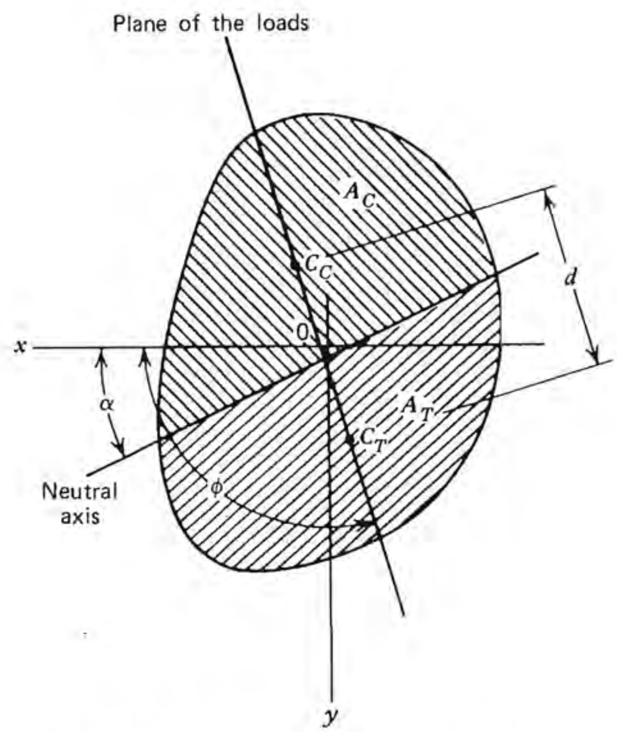


Fig. 6-5.1/Location of a neutral axis for fully plastic bending of an unsymmetrically loaded beam.

The yield point stress Y is uniform over the area A_T that has yielded in tension; the resultant tensile force $P_T = YA_T$ is located at the centroid C_T of A_T. Similarly the resultant compressive force $P_C = YA_C$ is located at the centroid C_C of A_C. The fully plastic moment M_P is given by

$$M_P = YA_T d = \frac{YAd}{2} \tag{6-5.1}$$

where d is the distance between the centroids C_T and C_C as indicated in Fig. 6-5.1. A plane through the centroids C_T and C_C is the plane of the loads for the beam. In case the calculated angle ϕ (Fig. 6-5.1) does not correspond to the plane of the applied loads, a new value is assumed for α and the calculations are repeated. Once the angle ϕ (Fig. 6-5.1) corresponds to the plane of the applied loads, the magnitude of the fully plastic load is calculated by setting the moment of the applied loads equal to M_P given by Eq. (6-5.1).

EXAMPLE 6-5.1

Fully Plastic Moment for Unsymmetrical Bending

A steel beam has the cross section shown in Fig. E6-5.1. The beam is made of a steel having a yield point stress $Y = 280$ MPa. Determine the fully plastic moment for the condition that the neutral axis passes through point B. Locate the resulting neutral axis and the plane of the loads.

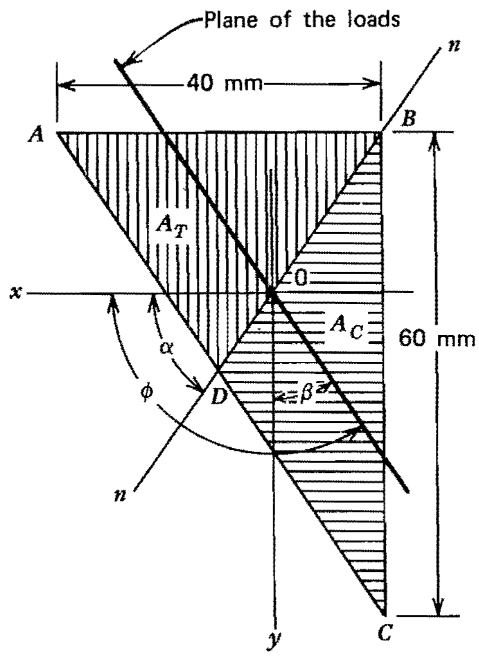


Fig. E6-5.1

SOLUTION

The neutral axis must divide the cross section into two equal areas since the area that has yielded in tension A_T must equal the area that has yielded in compression A_C. The neutral axis extends from point B to point D, the midpoint of AC:

$$\tan\alpha = \frac{30}{20} = 1.5$$

$$\alpha = 0.9828 \text{ rad}$$

The plane of the loads passes through the centroids of areas ABD and BCD. The centroids of these areas are located at $(\frac{20}{3}, -10)$ for ABD and $(-\frac{20}{3}, 10)$ for BCD.

$$\tan\beta = \frac{20/3-(-20/3)}{10-(-10)} = 0.6667$$

$$\beta = 0.5880 \text{ rad}$$

$$\phi = \pi/2 + \beta = 2.1588 \text{ rad}$$

The fully plastic moment M_P is equal to the product of the force on either of the two areas (A_T or A_C) and the distance d between the two centroids.

$$d = \sqrt{\left(20 - \frac{20}{3}\right)^2 + (30-10)^2} = 24.04 \text{ mm}$$

$$M_P = A_T Y d = \tfrac{1}{2}(40)(30)(280)(24.04) = 4.039 \times 10^6 \text{ N}\cdot\text{mm}$$

$$= 4.039 \text{ kN}\cdot\text{m}$$

PROBLEM SET 6-5

1. The cantilever beam in Problem 6-2.11 is made of a low carbon steel that has a yield stress $Y = 200$ MPa. (a) Determine the fully plastic load P_P for the beam for the condition that $\alpha = 0$. (b) Determine the fully plastic load P_P for the beam for the condition that $\alpha = \pi/6$ rad.

2. The cantilever beam in Problem 6-2.13 is made of a mild steel that has a yield point stress $Y = 240$ MPa. Determine the fully plastic load P_P for the condition that $\alpha = 0$.

   Ans. $P_P = 21.21$ kN at $\phi = 1.2679$ rad

REFERENCES

1. H. L. Langhaar and A. P. Boresi, *Engineering Mechanics*, McGraw-Hill, New York, 1959.
2. A. P. Boresi and P. P. Lynn, *Elasticity in Engineering Mechanics*, Prentice-Hall, Englewood Cliffs, New Jersey, 1974.

Additional References

1. J. N. Goodier, "A Theorem on the Shearing Stress in Beams with Applications to Multicellular Sections," *Journal of the Aeronautical Sciences*, Vol. 11, No. 3, July 1944, pp. 272–280.
2. K. Washizu, "Some Considerations on the Center of Shear," *Transactions of Japan Society for Aeronautical and Space Sciences*, Vol. 9, No. 15, 1966, pp. 77–83.
3. A. Weinstein, "The Center of Shear and the Center of Twist," *Quarterly of Applied Mathematics*, Vol. 5, No. 1, 1947, pp. 97–99.

CHAPTER 8

CURVED BEAMS

8-1 INTRODUCTION

The flexure formula (Eq. 6-1.1) is accurate for symmetrically loaded straight beams subjected to pure bending. It is also generally used to obtain approximate results for the design of straight beams subjected to shear loads, when the plane of loads contains the shear center and is parallel to a principal axis of the beam; the resulting errors in the computed stresses are small enough to be negligible as long as the beam length is at least five times the maximum cross-sectional dimension. In addition, the flexure formula is used in the design of curved beams for which the radius of curvature is more than five times the beam depth for the same reason. However, for curved beams the error in the computed stress predicted by the flexure formula increases as the ratio of the radius of curvature of the beam to the depth of the beam decreases in magnitude. Hence, as this ratio decreases, one needs a more accurate solution for curved beams.

Timoshenko[1] has presented a solution based on the theory of elasticity for the linear elastic behavior of curved beams of rectangular cross sections for the loading shown in Fig. 8-1.1*a*. He used polar coordinates and obtained relations for the radial stress σ_{rr}, the circumferential stress $\sigma_{\theta\theta}$, and the shearing stress $\sigma_{r\theta}$ (Fig. 8-1.1*b*). However, most curved beams do not have rectangular cross sections. Therefore, in the following we present an approximate curved beam solution that is generally applicable to all symmetrical cross sections. This solution is based upon two simplifying assumptions: (1) plane sections before loading remain plane after loading and (2) the radial stress σ_{rr} and shearing stress $\sigma_{r\theta}$ are sufficiently small so that the state of stress is essentially one dimensional. The resulting formula for the circumferential stress $\sigma_{\theta\theta}$ is the curved beam formula.

8-2 CIRCUMFERENTIAL STRESS IN A CURVED BEAM

Consider the curved beam shown in Fig. 8-2.1*a*. The cross section of the beam has a plane of symmetry. We assume that the applied loads all lie

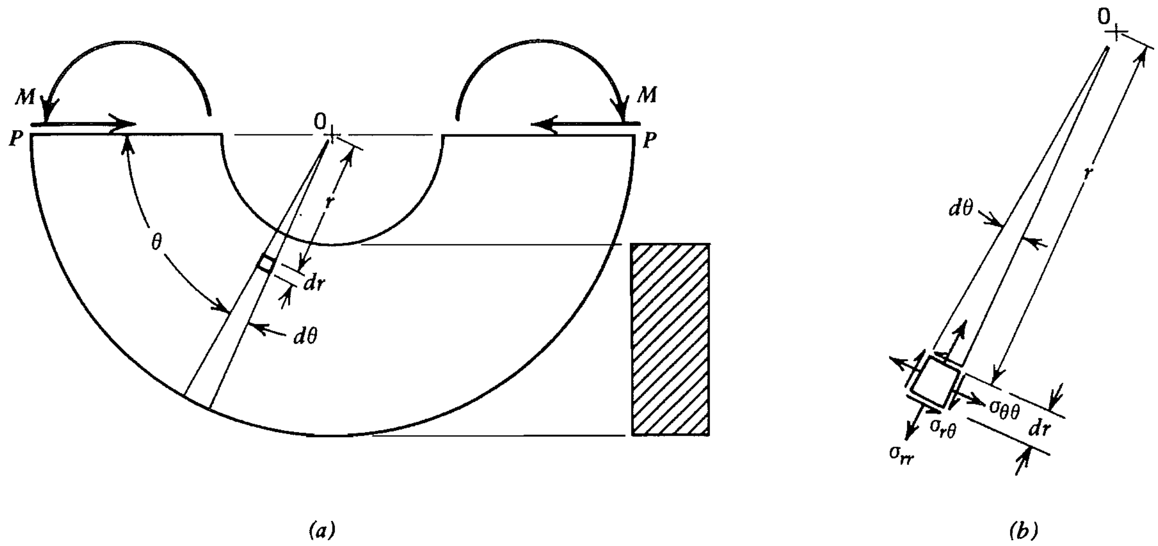


Fig. 8-1.1/Rectangular section curved beam. (*a*) Curved beam loading. (*b*) Stress components.

in one plane, which coincides with the plane of symmetry. The applied loads produce a positive moment, as shown in Fig. 8-2.1*b*, at each section of the curved beam. Thus, because of a positive moment, the radius of curvature, at each section of the beam, is increased in magnitude. We wish to determine an approximate formula for the circumferential stress distribution $\sigma_{\theta\theta}$ on section *BC*. A free body diagram of an element *FBCH* of the beam is shown in Fig. 8-2.1*b* (see Fig. 8-2.1*a*). The normal traction *N*, at the centroid of the cross section, the shear *V*, and moment M_x acting on face *FH* are shown in their positive directions. These forces must be balanced by the resultants due to the normal stress $\sigma_{\theta\theta}$ and the shearing stress $\sigma_{r\theta}$ that act on face *BC*. The effect of the shearing stress $\sigma_{r\theta}$ on the computation of $\sigma_{\theta\theta}$ is usually small, except for curved beams with very thin webs. However, since ordinarily, practical curved beams are not designed with thin webs because of the possibility of failure by excessive radial stresses (see Art. 8-3), in practice, neglecting the effect of $\sigma_{r\theta}$ on the computation of $\sigma_{\theta\theta}$ is reasonable.

Let the *z*-axis be normal to face *BC* (Fig. 8-2.1*b*). By equilibrium of forces in the *z*-direction and of moments about the centroidal *x*-axis, we find

$$\sum F_z = \int \sigma_{\theta\theta}\, dA - N = 0$$

$$\sum M_x = \int \sigma_{\theta\theta}(R-r)\, dA - M_x = 0$$

or

$$N = \int \sigma_{\theta\theta}\, dA \tag{8-2.1}$$

$$M_x = \int \sigma_{\theta\theta}(R-r)\, dA \tag{8-2.2}$$

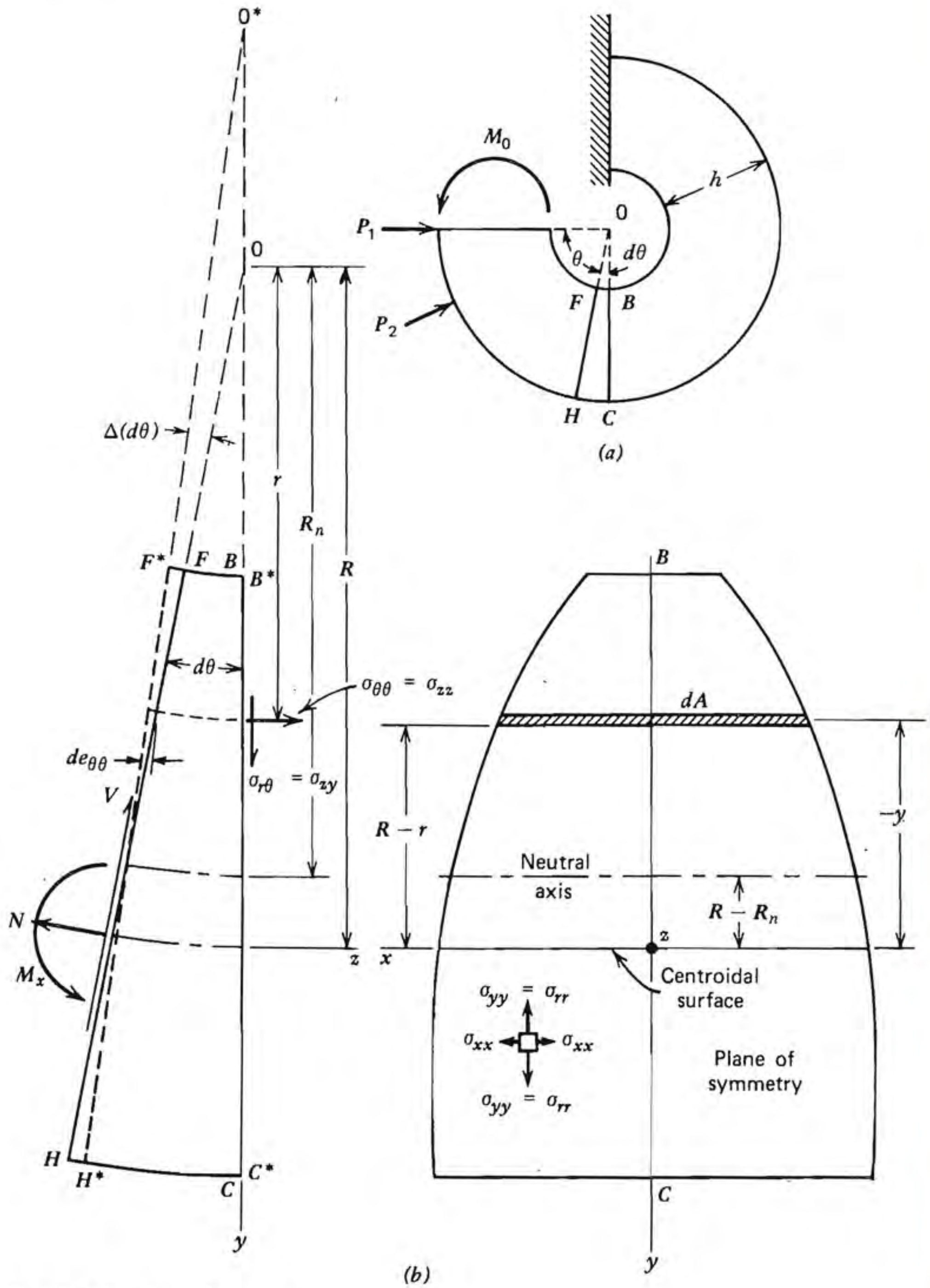


Fig. 8-2.1/Curved beam.

where R is the distance from the center of curvature of the curved beam to the centroid of the beam cross section and r locates the element of area dA from the center of curvature. The integrals of Eqs. (8-2.1) and (8-2.2) cannot be evaluated until $\sigma_{\theta\theta}$ is expressed in terms of r. The functional relationship between $\sigma_{\theta\theta}$ and r is obtained from the assumed geometry of deformation and the stress-strain relations for the material.

The curved beam element $FBCH$ in Fig. 8-2.1b represents the element in the undeformed state. The element $F^*B^*C^*H^*$ represents the element after it is deformed by the loads. For simplicity, we have positioned the deformed element so that face B^*C^* coincides with face BC. As in the case of straight beams, we assume that plane B^*C^* remains plane under the deformation. Face F^*H^* of the deformed curved beam element forms an angle $\Delta(d\theta)$ with respect to FH. Line F^*H^* intersects line FH at the neutral axis of the cross section (axis for which $\sigma_{\theta\theta} = 0$) at distance R_n from the center of curvature. The movement of the center of curvature from point 0 to point 0* is exaggerated in Fig. 8-2.1b in order to visualize the geometry changes. For infinitesimally small displacements, the movement of the center of curvature is infinitesimal. The elongation $de_{\theta\theta}$ of a typical element in the θ direction is equal to the distance between faces FH and F^*H^* and varies linearly with the distance $(R_n - r)$. The corresponding strain $\epsilon_{\theta\theta}$ however, is a nonlinear function of r, since the element length $r\,d\theta$ varies with r. This fact distinguishes a curved beam from a straight beam. Thus, by Fig. 8-2.1b, we obtain for the strain

$$\epsilon_{\theta\theta} = \frac{de_{\theta\theta}}{r\,d\theta} = \frac{(R_n - r)\,\Delta(d\theta)}{r\,d\theta} = \left(\frac{R_n}{r} - 1\right)\omega \tag{8-2.3}$$

where

$$\omega = \frac{\Delta(d\theta)}{d\theta} \tag{8-2.4}$$

It is assumed that σ_{xx} is sufficiently small so that it may be discarded. Hence, the curved beam is considered to be a problem in plane stress. Although radial stress σ_{rr} may, in certain cases, be of importance (see Art. 8-3), here we neglect its effect on $\epsilon_{\theta\theta}$. Then, by Hooke's law, we find

$$\sigma_{\theta\theta} = E\epsilon_{\theta\theta} = \frac{R_n - r}{r}E\omega = \frac{E\omega R_n}{r} - E\omega \tag{8-2.5}$$

Substituting Eq. (8-2.5) into Eqs. (8-2.1) and (8-2.2), we obtain

$$N = R_n E\omega \int \frac{dA}{r} - E\omega \int dA = R_n E\omega A_m - E\omega A \tag{8-2.6}$$

$$\begin{aligned} M_x &= R_n R E\omega \int \frac{dA}{r} - (R + R_n)E\omega \int dA + E\omega \int r\,dA \\ &= R_n R E\omega A_m - (R + R_n)E\omega A + E\omega R A = R_n E\omega(RA_m - A) \end{aligned} \tag{8-2.7}$$

where A is the cross sectional area of the curved beam and A_m has the dimensions of length and is defined by the relation

$$A_m = \int \frac{dA}{r} \tag{8-2.8}$$

Equation (8-2.7) can be rewritten in the form

$$R_n E\omega = \frac{M_x}{RA_m - A} \tag{8-2.9}$$

Then substitution into Eq. (8-2.6) gives

$$E\omega = \frac{A_m M_x}{A(RA_m - A)} - \frac{N}{A} \tag{8-2.10}$$

The circumferential stress distribution for the curved beam is obtained by substituting Eq. (8-2.9) and (8-2.10) into Eq. (8-2.5) to obtain the curved beam formula

$$\sigma_{\theta\theta} = \frac{N}{A} + \frac{M_x(A - rA_m)}{Ar(RA_m - A)} \tag{8-2.11}$$

The normal stress distribution given by Eq. (8-2.11) is hyperbolic in form; that is, it varies as $1/r$. For the case of a curved beam with rectangular cross section ($R/h = 0.75$) subjected to pure bending, the normal stress distribution is shown in Fig. 8-2.2.

Since Eq. (8-2.11) has been based on several simplifying assumptions, it is essential that its validity be verified. Results predicted by the curved beam formula can be compared with those obtained from the elasticity solution for curved beams with rectangular sections and with those obtained from the experiments on curved beams with other kinds of cross sections. The maximum value of the circumferential stress $\sigma_{\theta\theta(CB)}$ as given by the curved beam formula, may be computed from Eq. (8-2.11) for curved beams of rectangular cross sections subjected to pure bending and to shear (Fig. 8-2.3). The ratios of $\sigma_{\theta\theta(CB)}$ to the elasticity solution

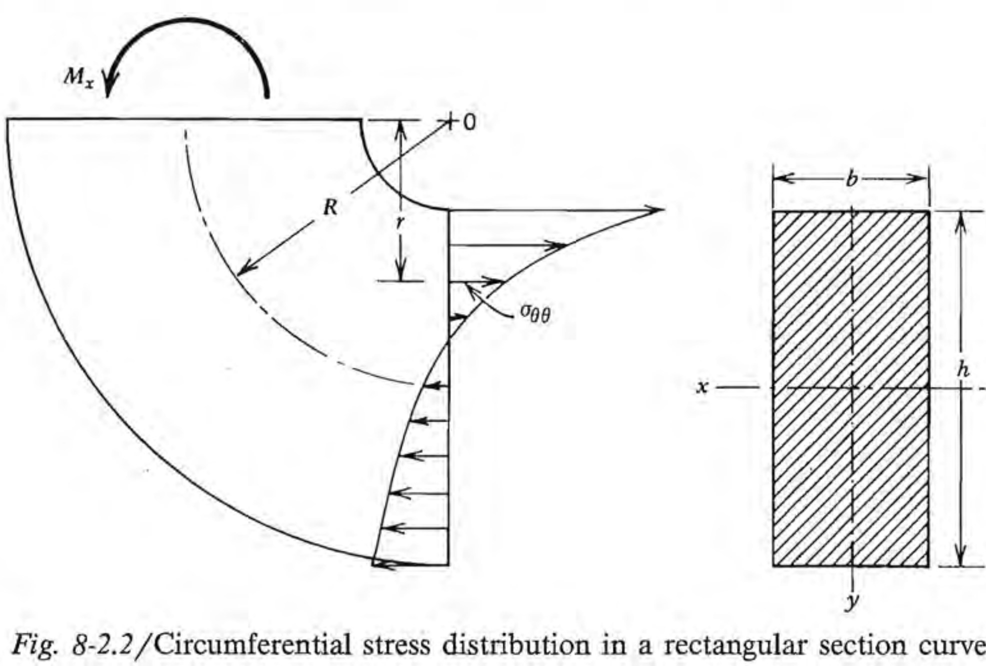


Fig. 8-2.2/Circumferential stress distribution in a rectangular section curved beam ($R/h = 0.75$).

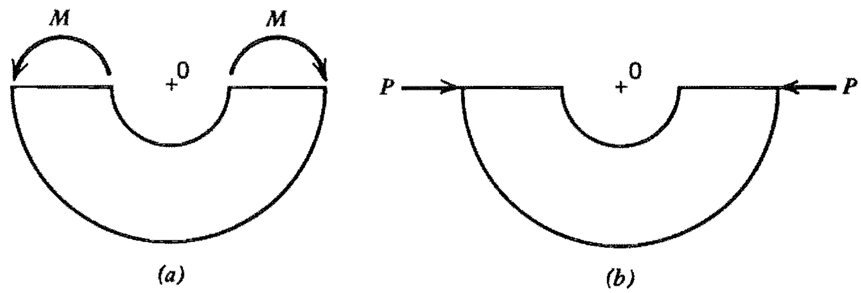


Fig. 8-2.3/Types of curved beam loadings. (*a*) Pure bending. (*b*) Shear loading.

$\sigma_{\theta\theta(\text{elast})}$ are listed in Table 8-2.1 for pure bending (Fig. 8-2.3*a*) and for shear loading (Fig. 8-2.3*b*), for several values of R/h, where h denotes the beam depth (Fig. 8-2.1*a*). The nearer these ratios are to one, the less error in Eq. (8-2.11). The curved beam formula is more accurate for pure bending than for shear loading. Most curved beams are subjected to a combination of bending and shear. The value of R/h is usually greater than 1.0 for curved beams, so that the error in the curved beam formula is not particularly significant. However, possible errors occur in the curved beam formula for I-section and T-section curved beams. These errors are discussed in Art. 8-4. Also listed in Table 8-2.1 are the ratios of the maximum circumferential stress $\sigma_{\theta\theta(st)}$ given by the straight beam flexure formula (Eq. 6-1.1) to the value $\sigma_{\theta\theta(\text{elast})}$. The straight beam solution is appreciably in error for small values of R/h and is in error by 7 percent for $R/h = 5.0$; the error is nonconservative. Generally, for curved beams with R/h greater than 5.0, the flexure formula is used.

As R becomes large compared to h, the right-hand term in Eq. (8-2.11) reduces to $-M_x y/I_x$. The negative sign results because the sign conven-

Table 8-2.1

Ratios of the Maximum Circumferential Stress in Rectangular Section Curved Beams as Computed by Elasticity Theory, by the Curved Beam Formula and by the Flexure Formula

	Pure Bending		*Shear Loading*	
$\frac{R}{h}$	$\frac{\sigma_{\theta\theta(CB)}}{\sigma_{\theta\theta(elast)}}$	$\frac{\sigma_{\theta\theta(st)}}{\sigma_{\theta\theta(elast)}}$	$\frac{\sigma_{\theta\theta(CB)}}{\sigma_{\theta\theta(elast)}}$	$\frac{\sigma_{\theta\theta(st)}}{\sigma_{\theta\theta(elast)}}$
0.65	1.046	0.439	0.855	0.407
0.75	1.012	0.526	0.898	0.511
1.0	0.997	0.654	0.946	0.653
1.5	0.996	0.774	0.977	0.776
2.0	0.997	0.831	0.987	0.834
3.0	0.999	0.888	0.994	0.890
5.0	0.999	0.933	0.998	0.934

tion for moments for curved beams is opposite to that for straight beams (see Eq. 6-1.1). To prove this reduction, note that $r = R + y$. Then the term RA_m in Eq. (8-2.11) may be written as

$$RA_m = \int \left(\frac{R}{R+y} + 1 - 1 \right) dA = A - \int \frac{y}{R+y}\, dA \qquad \text{(a)}$$

Hence, the denominator of the right-hand term in Eq. (8-2.11) becomes, for $R/h \to \infty$,

$$\begin{aligned} Ar(RA_m - A) &= -A \int \left(\frac{Ry}{R+y} + y - y \right) dA - Ay \int \frac{y}{R+y}\, dA \\ &= A \int \frac{y^2}{R+y}\, dA - A \int y\, dA - Ay \int \frac{y}{R+y}\, dA \\ &= \frac{A}{R} \int \frac{y^2}{1+\frac{y}{R}}\, dA - A \int y\, dA - \frac{Ay}{R} \int \frac{y}{1+\frac{y}{R}}\, dA \\ &= \frac{AI_x}{R} \end{aligned} \qquad \text{(b)}$$

since as $R/h \to \infty$, $y/R \to 0$, $1 + y/R \to 1$, $\int [y^2\, dA/(1+y/R)] \to I_x$, and $\int [y\, dA/(1+y/R)] \to 0$. The right-hand term in Eq. (8-2.11) then

Table 8-2.2

Analytical Expressions for A, R, and $A_m = \int \frac{dA}{r}$

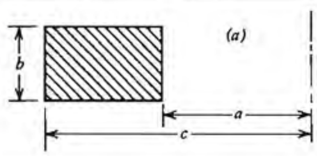
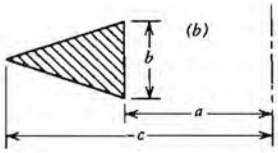
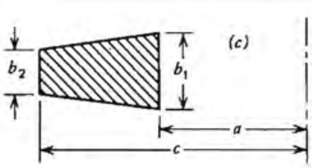
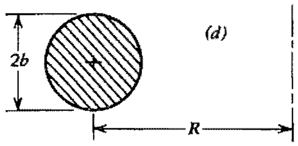
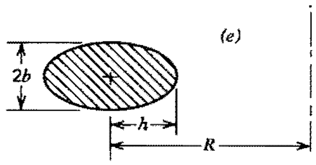
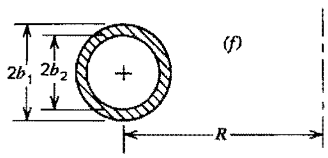
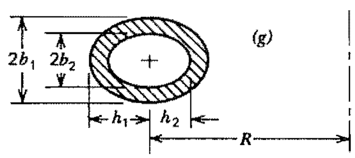
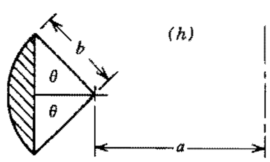
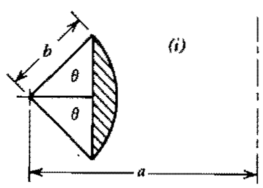
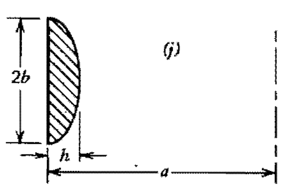
	$A = b(c-a)$; $A_m = b \ln \frac{c}{a}$	$R = \frac{a+c}{2}$
	$A = \frac{b}{2}(c-a)$; $A_m = \frac{bc}{c-a} \ln \frac{c}{a} - b$	$R = \frac{2a+c}{3}$
	$A = \frac{b_1+b_2}{2}(c-a)$; $A_m = \frac{b_1 c - b_2 a}{c-a} \ln \frac{c}{a} - b_1 + b_2$	$R = \frac{a(2b_1+b_2)+c(b_1+2b_2)}{3(b_1+b_2)}$

Table 8-2.2

Continued

	$A=\pi b^2$ $A_m=2\pi(R-\sqrt{R^2-b^2})$
	$A=\pi bh$ $A_m=\dfrac{2\pi b}{h}(R-\sqrt{R^2-h^2})$
	$A=\pi(b_1{}^2-b_2{}^2)$ $A_m=2\pi\left(\sqrt{R^2-b_2{}^2}-\sqrt{R^2-b_1{}^2}\right)$
	$A=\pi(b_1h_1-b_2h_2)$ $A_m=2\pi\left(\dfrac{b_1R}{h_1}-\dfrac{b_2R}{h_2}-\dfrac{b_1}{h_1}\sqrt{R^2-h_1{}^2}+\dfrac{b_2}{h_2}\sqrt{R^2-h_2{}^2}\right)$
	$A=b^2\theta-\dfrac{b^2}{2}\sin 2\theta; R=a+\dfrac{4b\sin^3\theta}{3(2\theta-\sin 2\theta)}$. For $a>b$, $A_m=2a\theta-2b\sin\theta-\pi\sqrt{a^2-b^2}+2\sqrt{a^2-b^2}\sin^{-1}\left[\dfrac{b+a\cos\theta}{a+b\cos\theta}\right]$ For $b>a$, $A_m=2a\theta-2b\sin\theta+2\sqrt{b^2-a^2}\ln\left[\dfrac{b+a\cos\theta+\sqrt{b^2-a^2}\sin\theta}{a+b\cos\theta}\right]$
	$A=b^2\theta-\dfrac{b^2}{2}\sin 2\theta;\ R=a-\dfrac{4b\sin^3\theta}{3(2\theta-\sin 2\theta)}$ $A_m=2a\theta+2b\sin\theta-\pi\sqrt{a^2-b^2}-2\sqrt{a^2-b^2}\sin^{-1}\left[\dfrac{b-a\cos\theta}{a-b\cos\theta}\right]$
	$A=\dfrac{\pi bh}{2};\ R=a-\dfrac{4h}{3\pi}$ $A_m=2b+\dfrac{\pi b}{h}(a-\sqrt{a^2-h^2})-\dfrac{2b}{h}\sqrt{a^2-h^2}\sin^{-1}\left(\dfrac{h}{a}\right)$

simplifies to

$$\frac{M_x R}{AI_x}(A - RA_m - yA_m) = \frac{M_x R}{AI_x}\left(\int \frac{y/R}{1+\frac{y}{R}}\,dA - \frac{y}{R}\int \frac{dA}{1+\frac{y}{R}}\right) = -\frac{M_x y}{I_x} \tag{c}$$

The curved beam solution for curved beams (see Eq. 8-2.11) requires that A_m defined by Eq. (8-2.8) be calculated for cross sections of various shapes. The number of significant digits retained in calculating A_m must be greater than that required for $\sigma_{\theta\theta}$ since RA_m approaches the value of A as R/h becomes large (see Eq. a). Explicit formulas for A, A_m, and R for several curved beam cross sectional areas are listed in Table 8-2.2. Often the cross section of a curved beam is composed of two or more of the fundamental areas listed in Table 8-2.2. The values of A, A_m, and R for the composite area are given by summation. Thus, for composite cross sections,

$$A = \sum_{i=1}^{n} A_i \tag{8-2.12}$$

$$A_m = \sum_{i=1}^{n} A_{m_i} \tag{8-2.13}$$

$$R = \frac{\sum_{i=1}^{n} R_i A_i}{\sum_{i=1}^{n} A_i} \tag{8-2.14}$$

where n is the number of fundamental areas that form the composite area.

Location of Neutral Axis of Cross Section/The neutral axis of bending of the cross section is defined by the conditions $\sigma_{\theta\theta} = 0$. The neutral axis is located at distance R_n from the center of curvature. The magnitude of R_n is obtained from Eq. (8-2.11) with the condition that $\sigma_{\theta\theta} = 0$ on the neutral surface $r = R_n$. Thus, Eq. (8-2.11) yields

$$R_n = \frac{AM_x}{A_m M_x + N(A - RA_m)} \tag{8-2.15}$$

For pure bending, $N = 0$, and then Eq. (8-2.15) yields

$$R_n = \frac{A}{A_m} \tag{8-2.16}$$

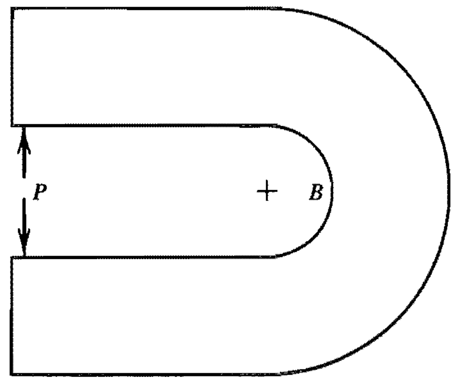


Fig. E8-2.1

EXAMPLE 8-2.1

Stress in Curved Beam Portion of a Frame

The frame shown in Fig. E8-2.1 has a square cross section with dimensions of 50.0 mm. The load P is located 100 mm from the center of curvature of the curved beam portion of the frame. The radius of curvature of the inner surface of the curved beam is $a = 30$ mm. If $P = 9.50$ kN, determine the values for the maximum tensile and compressive stresses in the frame.

SOLUTION

The circumferential stresses $\sigma_{\theta\theta}$ are calculated using Eq. (8-2.11). Required values for A, A_m, and R for the curved beam are calculated using the equations in Row a of Table 8-2.2. For the curved beam $a = 30$ mm and $c = 80$ mm.

$$A = b(c - a) = 50(80 - 30) = 2500 \text{ mm}^2$$

$$A_m = b \ln \frac{c}{a} = 50 \ln \frac{80}{30} = 49.04 \text{ mm}$$

$$R = \frac{a + c}{2} = \frac{80 + 30}{2} = 55 \text{ mm}$$

Hence, the maximum tensile stress is

$$\sigma_{\theta\theta B} = \frac{P}{A} + \frac{M_x(A - rA_m)}{Ar(RA_m - A)} = \frac{9500}{2500} + \frac{155(9500)[2500 - 30(49.04)]}{2500(30)[55(49.04) - 2500]}$$

$$= 106.2 \text{ MPa}$$

The maximum compressive stress is

$$\sigma_{\theta\theta C} = \frac{9500}{2500} + \frac{155(9500)[2500 - 80(49.04)]}{2500(80)[55(49.04) - 2500]} = -49.3 \text{ MPa}$$

EXAMPLE 8-2.2
Stresses in a Crane Hook

Section BC is the critically stressed section of a crane hook (Fig. E8-2.2). For a large number of manufactured crane hooks, the critical section BC can be closely approximated by a trapezoidal area with half of an ellipse at the inner radius and an arc of a circle at the outer radius. Such a section is shown in Fig. E8-2.2b, including dimensions for the critical cross section. The crane hook is made of a ductile steel that has a yield stress of $Y = 500$ MPa. Assuming that the crane hook is to be designed with a factor of safety of $SF = 2.00$ against initiation of yielding, determine the maximum load P that can be carried by the crane hook.

SOLUTION

The circumferential stresses $\sigma_{\theta\theta}$ are calculated using Eq. (8-2.11). To calculate values of A, R, and A_m for the curved beam cross section, we divide the cross section into basic areas A_1, A_2, and A_3 (Fig. E8-2.2b).

For area A_1, $a = 84$ mm. Substituting this dimension along with other given dimensions into Table 8-2.2, row j, we find

$$A_1 = 1658.76 \text{ mm}^2 \qquad R_1 = 73.81 \text{ mm} \qquad A_{m_1} = 22.64 \text{ mm} \qquad (1)$$

For the trapezoidal area A_2, $a = 60 + 24 = 84$ mm and $c = a + 100 = 184$ mm. Substituting these dimensions along with other given dimensions into Table 8-2.2, row c, we find

$$A_2 = 6100.00 \text{ mm}^2 \qquad R_2 = 126.62 \text{ mm} \qquad A_{m_2} = 50.57 \text{ mm} \qquad (2)$$

For area A_3, $\theta = 0.5721$ rad, $b = 31.40$ mm, and $a = 157.60$ mm. When these values are substituted into Table 8-2.2, row h, we obtain

$$A_3 = 115.27 \text{ mm}^2 \qquad R_3 = 186.01 \text{ mm} \qquad A_{m_3} = 0.62 \text{ mm} \qquad (3)$$

Substituting values of A_i, R_i, and A_{mi} from Eqs. (1), (2), and (3) into Eqs. (8-2.12), (8-2.13), and (8-2.14), we calculate

$$A = 6100.00 + 115.27 + 1658.76 = 7874.03 \text{ mm}^2$$
$$A_m = 50.57 + 0.62 + 22.64 = 73.83 \text{ mm}$$
$$R = \frac{6100.00(126.62) + 115.27(186.01) + 1658.76(73.81)}{7874.03}$$
$$= 116.37 \text{ mm}$$

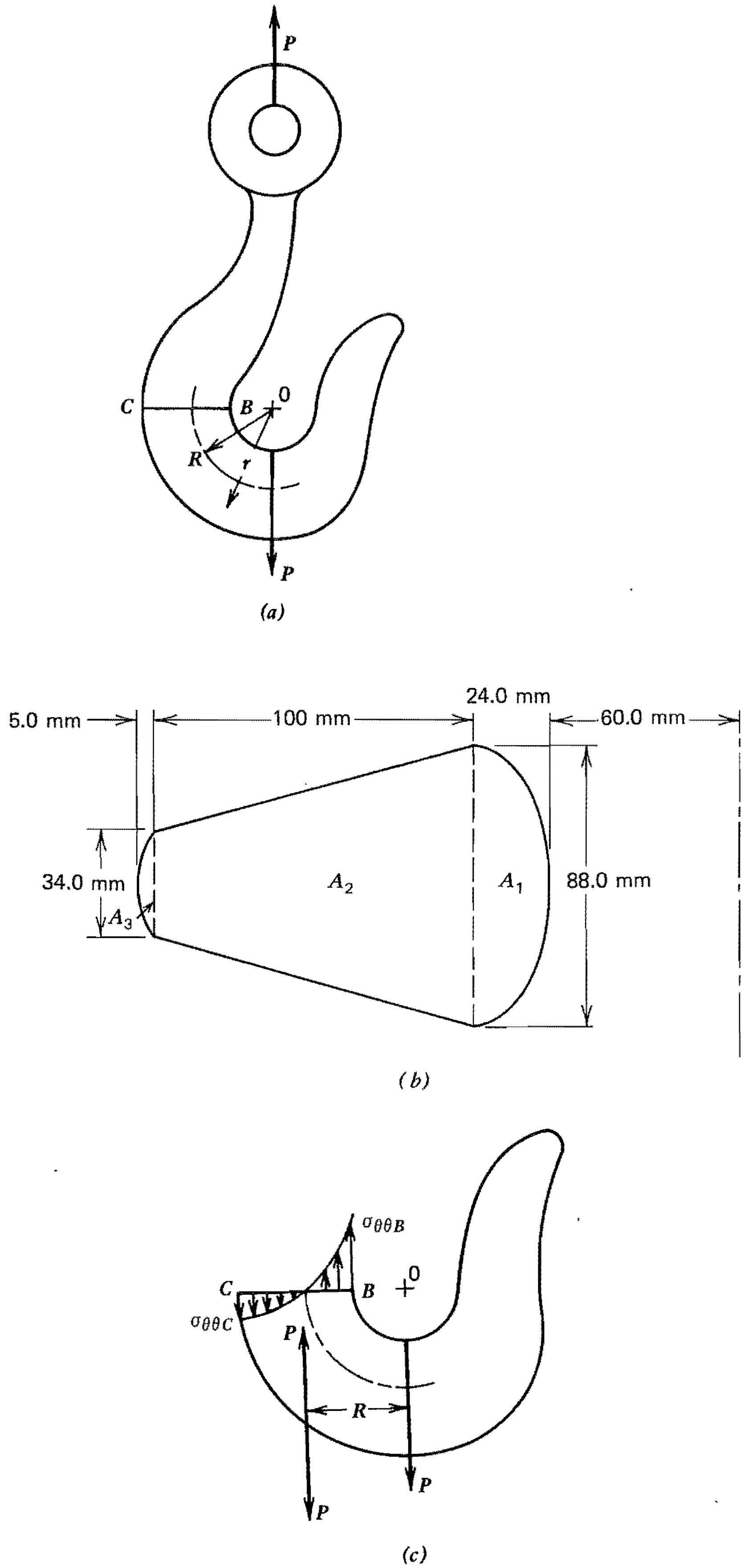


Fig. E8-2.2/Crane hook.

As indicated in Fig. E8-2.2c, the circumferential stress distribution $\sigma_{\theta\theta}$ is due to the normal load $N=P$ and the moment $M_x = PR$. The maximum tension and compression values of $\sigma_{\theta\theta}$ occur at points B and C, respectively. For points B and C, Fig. E8-2.2b yields

$$r_B = 60 \text{ mm}$$

$$r_C = 60 + 24 + 100 + 5 = 189 \text{ mm}$$

Substituting the required values into Eq. (8-2.11), we find

$$\sigma_{\theta\theta B} = \frac{P}{7874.03} + \frac{116.37P[7874.03 - 60(73.83)]}{7874.03(60)[116.37(73.83) - 7874.03]}$$

$$= 0.000127P + 0.001182P$$

$$= 0.001309P \text{ (tension)}$$

$$\sigma_{\theta\theta C} = \frac{P}{7874.03} + \frac{116.37P[7874.03 - 189(73.83)]}{7874.03(189)[116.37(73.83) - 7874.03]}$$

$$= 0.000127P - 0.000662P$$

$$= -0.000535P \text{ (compression)}$$

Since the absolute magnitude of $\sigma_{\theta\theta B}$ is greater than $\sigma_{\theta\theta C}$ initiation of yield occurs when $\sigma_{\theta\theta B}$ equals the yield stress Y. The corresponding value of the failure load (P_f) is the load at which yield occurs. Dividing the failure load $P_f = Y/(0.001309)$ by the factor of safety $SF = 2.00$, we obtain the design load P; namely,

$$P = \frac{500}{2.00(0.001309)} = 190{,}900 \text{ N}$$

Thus, based upon a factor of safety of 2.00 at sea level on earth, the crane hook can support a weight of mass $m = P/g$, where g denotes the acceleration due to gravity. In other words, a mass as large as $m = 190{,}900/9.81 = 19{,}460$ kg, can be hoisted by the crane with a factor of safety of 2.00 against yield initiation.

Computer Program for Crane Hooks / To expedite the solution for crane hooks, a FORTRAN computer program CRAHK is presented in Table E8-2.2, along with a set of input data and the corresponding output. As noted from the program and data, several cases can be calculated with one computer run. Consequently, parametric studies of the crane hook dimensions can be rapidly performed.

Table E8-2.2

FORTRAN Program for Crane Hooks

```
        PROGRAM CRAHK(INPUT,OUTPUT,CRIN,CROUT,TAPE5 =
      ACRIN,TAPE6 = CROUT)
        READ (5,*) NN
        DO 2 J = 1,NN
        WRITE (6,22)
 22     FORMAT ("      H1      H2      H3      B1      B2")
        READ (5,*) N,EH1,EH2,EH3,B1,B2
        WRITE (6,23) EH1,EH2,EH3,B1,B2
 23     FORMAT (5F10.2)
        WRITE (6,24)
 24     FORMAT ("      RB      R      AREA      AM"
      C"    SIG(MAX)/P   SIG(MIN)/P")
        DO 3 I = 1,N
        READ (5,*) RB
        A1 = .7854*B1*EH1
        AA = RB + EH1
        C = AA + EH2
        R1 = AA - .42441*EH1
        D = SQRT(AA**2 - EH1**2)
        AM1 = B1 + 1.5708*B1*(AA - D)/EH1 - B1*D* ASIN(EH1/AA)
      A/EH1
        A2 = .5*(B1 + B2)*EH2
        R2 = (AA*(2.*B1 + B2) + C*(B1 + 2.*B2))/(3.*(B1 + B2))
        AM2 = (B1*C - B2*AA)*ALOG(C/AA)/EH2 - B1 + B2
        B3 = (B2**2 + EH3**2*4.)/(8.*EH3)
        THETA = ASIN(.5*B2/B3)
        A3 = B3**2*THETA - .5*B3**2*SIN(2.*THETA)
        A = C + EH3 - B3
        R3 = A + 1.3333*B3*SIN(THETA)**3/(2.*THETA - SIN(2.*
      ATHETA))
        DD = SQRT(A**2 - B3**2)
        Q = (B3 + A*COS(THETA))/(A + B3*COS(THETA))
        AM3 = 2.*A*THETA - 2.*B3*SIN(THETA) - 3.14159*DD +
      A2.*DD*ASIN(Q)
        AREA = A1 + A2 + A3
        AM = AM1 + AM2 + AM3
        R = (R1*A1 + R2*A2 + R3*A3)/AREA
        SIG1 = 1./AREA + R*(AREA - RB*AM)/(AREA*RB*(R*AM
      A - AREA))
        SIG2 = 1./AREA + R*(AREA - (C + EH3)*AM)/(AREA*(C +
      AEH3)*(R*AM - AREA))
        WRITE (6,25) RB,R,AREA,AM,SIG1,SIG2
 25     FORMAT (4F12.2,2F12.7)
 3      CONTINUE
```

Table 8-2.2

Continued

```
    2   CONTINUE
        STOP
        END
```

INPUT FOR PROGRAM CRAHK

```
3
1,63.,303.,28.33,184.,74.
165.
5,.93,3.97,.18,3.46,1.35
2.25
2.375
2.5
2.675
2.75
5,.83,4.07,.18,3.50,1.35
2.25
2.375
2.50
2.675
2.75
```

OUTPUT FOR PROGRAM CRAHK

H1	H2	H3	B1	B2	
63.00	303.00	28.33	184.00	74.00	
RB	R	AREA	AM	SIG(MAX)/P	SIG(MIN)/P
165.00	335.05	49741.25	163.95	0.0001986	−0.0000773
H1	H2	H3	B1	B2	
0.93	3.97	0.18	3.46	1.35	
RB	R	AREA	AM	SIG(MAX)/P	SIG(MIN)/P
2.25	4.47	12.24	3.00	0.8336250	−0.3283893
2.38	4.60	12.24	2.90	0.8407182	−0.3437211
2.50	4.72	12.24	2.81	0.8483529	−0.3590766
2.68	4.90	12.24	2.70	0.8598112	−0.3806141
2.75	4.97	12.24	2.65	0.8649603	−0.3898584
H1	H2	H3	B1	B2	
0.83	4.07	0.18	3.50	1.35	
RB	R	AREA	AM		SIG(MIN)/P
2.25	4.46	12.32	3.03	0.8118828	−0.3236749
2.38	4.58	12.32	2.93	0.8189578	−0.3388556
2.50	4.71	12.32	2.84	0.8265510	−0.3540606
2.68	4.88	12.32	2.72	0.8379177	−0.3753882
2.75	4.96	12.32	2.68	0.8430174	−0.3845428

PROBLEM SET 8-2

1. The frame shown in Fig. E8-2.1 has a rectangular cross section with a width of 10 mm and a depth of 40 mm. The load P is located 120 mm from the centroid of section BC. The frame is made of a steel having a yield stress of $Y = 430$ MPa. The frame has been designed using a factor of safety of $SF = 1.75$ against initiation of yielding. Determine the maximum allowable magnitude of P, if the radius of curvature at section BC is $R = 40$ mm.

2. Solve Problem 1 for the condition that $R = 35$ mm.

   Ans. $P = 3.174$ kN

3. The curved beam in Fig. P8-2.3 has a circular cross section 50 mm in diameter. The inside diameter of the curved beam is 40 mm. Determine the stress at B when $P = 20$ kN.

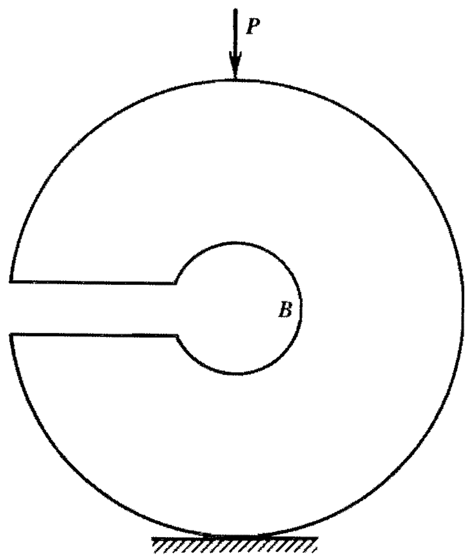


Fig. P8-2.3

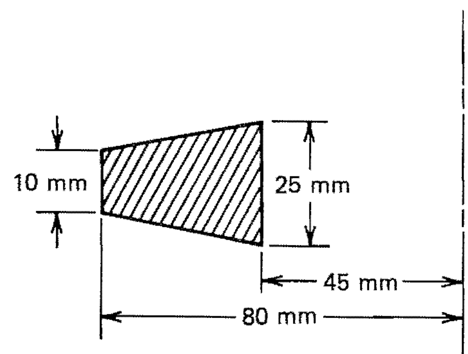


Fig. P8-2.4

4. Let the crane hook in Fig. E8-2.2 have a trapezoidal cross section as shown in Row c of Table 8-2.2 with (see Fig. P8-2.4) $a = 45$ mm, $c = 80$ mm, $b_1 = 25$ mm, and $b_2 = 10$ mm. Determine the maximum load to be carried by the hook if the working stress is 150 MPa.

   Ans. $P = 7.34$ kN

5. A curved beam is built up by welding rectangular and elliptical cross section curved beams together; the cross section is shown in Fig. P8-2.5. The center of curvature is located 20 mm from B. The curved

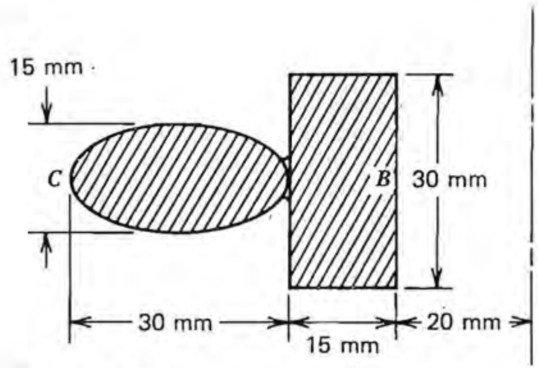


Fig. P8-2.5

beam is subjected to a positive bending moment M_x(N · m). Determine the stresses at points B and C in terms of M_x.

6. A commercial crane hook has the cross-sectional dimensions shown in Fig. P8-2.6 at the critical section that is subjected to an axial load $P = 100$ kN. Determine the circumferential stresses at the inner and outer radii for this load. Assume that area A_1 is half of an ellipse (see Row j in Table 8-2.2).

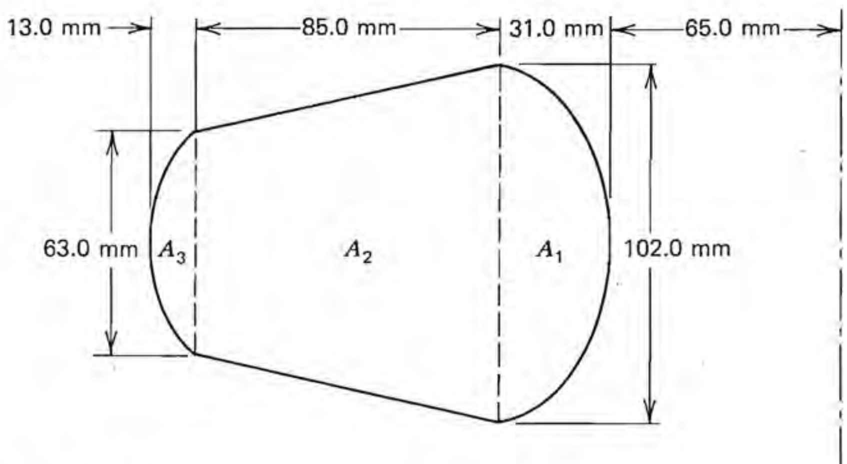


Fig. P8-2.6

Ans. $\sigma_{\theta\theta B} = 113.5$ MPa, $\sigma_{\theta\theta C} = -43.6$ MPa

7. A crane hook has the cross-sectional dimensions shown in Fig. P8-2.7 at the critical section that is subjected to an axial load $P = 90.0$ kN. Determine the circumferential stresses at the inner and outer radii for this load. Note that A_1 and A_3 are enclosed by circular arcs.

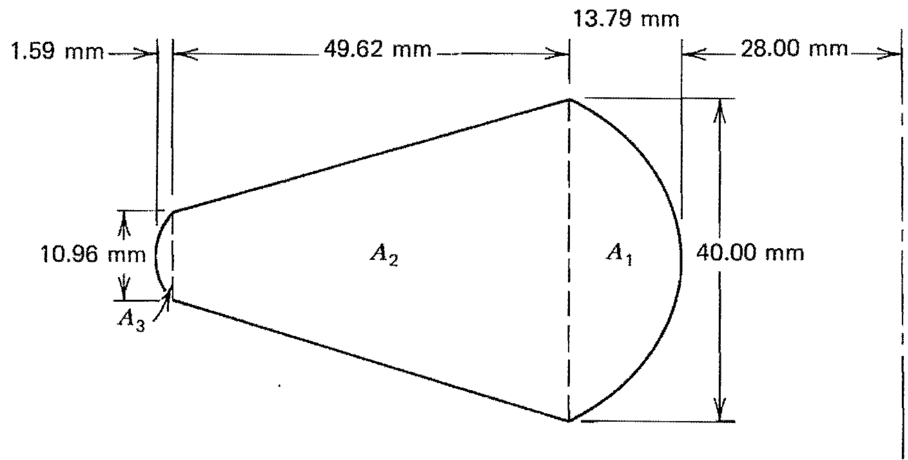


Fig. P8-2.7

8. The curved beam in Fig. P8-2.8 has a triangular cross section with the dimensions shown. If $P = 40$ kN, determine the circumferential stresses at B and C.

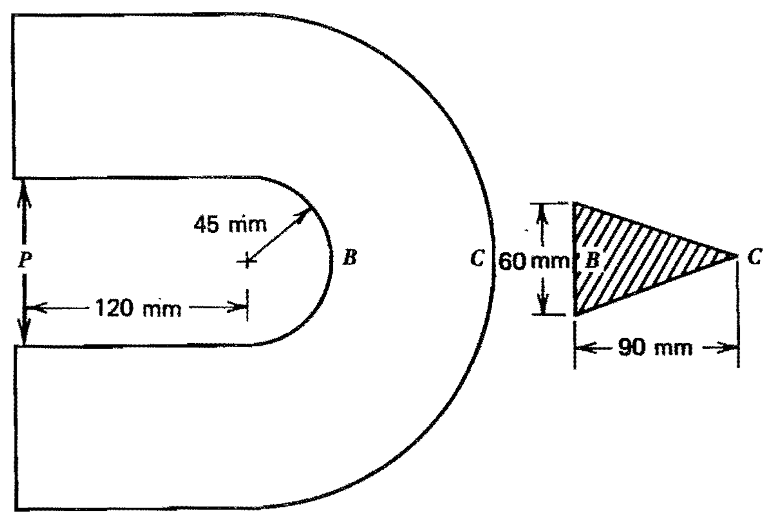


Fig. P8-2.8

Ans. $\sigma_{\theta\theta B} = 297.8$ MPa, $\sigma_{\theta\theta C} = -238.1$ MPa

8-3 RADIAL STRESSES IN CURVED BEAMS

The curved beam formula for circumferential stress $\sigma_{\theta\theta}$ (Eq. 8-2.11) is based on the assumption that the effect of radial stress is negligible. This assumption is quite accurate for curved beams with circular, rectangular, or trapezoidal cross sections; that is, cross sections that do not possess thin webs. However, in curved beams with cross sections in the form of

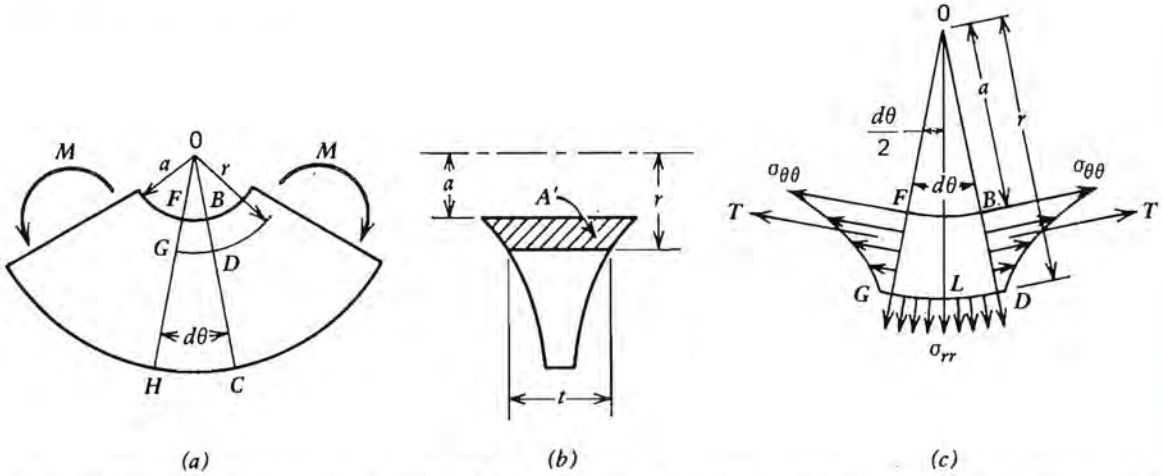


Fig. 8-3.1/Radial stress in a curved beam. (*a*) Side view. (*b*) Cross section shape. (*c*) Element *BDGF*.

an *H*, *T*, or *I*, the webs may be so thin that deformation of the cross section may produce a maximum radial stress in the web that may exceed the maximum circumferential stress. The beam should be designed so that this condition does not occur.

To illustrate the above remarks, we consider the radial stress that occurs in a curved beam at radius r from the center of curvature 0 of the beam (Fig. 8-3.1*a*). Consider equilibrium of the element *BDGF* of the beam shown enlarged in the free body diagram in Fig. 8-3.1*c*. The faces *BD* and *GF*, which subtend the infinitesimal angle $d\theta$, have the area A' shown shaded in Fig. 8-3.1*b*. The distribution of $\sigma_{\theta\theta}$ on each of these areas produces a resultant circumferential force T (Fig. 8-3.1*c*) given by the expression

$$T = \int_a^r \sigma_{\theta\theta}\, dA \tag{8-3.1}$$

The components of the circumferential forces along line $0L$ are balanced by the radial stress σ_{rr} acting on the area $tr\,d\theta$, where t is the thickness of the curved beam at the distance r from the center of curvature 0 (Fig. 8-3.1*b*). Thus for equilibrium in the radial direction along $0L$, $\Sigma F_r = 0 = \sigma_{rr} tr\, d\theta - 2T \sin(d\theta/2) = (\sigma_{rr} tr - T)\, d\theta$, since for infinitesimal angle $d\theta/2, \sin(d\theta/2) = d\theta/2$. Thus,

$$\sigma_{rr} = \frac{T}{tr} \tag{8-3.2}$$

The force T is obtained by substitution of Eq. (8-2.11) into Eq. (8-3.1). Thus,

$$T = \frac{N}{A}\int_a^r dA + \frac{M_x}{RA_m - A}\int_a^r \frac{dA}{r} - \frac{M_x A_m}{A(RA_m - A)}\int_a^r dA$$

$$T = \frac{A'}{A}N + \frac{AA'_m - A'A_m}{A(RA_m - A)}M_x \tag{8-3.3}$$

where

$$A'_m = \int_a^r \frac{dA}{r} \qquad \text{and} \qquad A' = \int_a^r dA \tag{8-3.4}$$

Substitution of Eq. (8-3.3) into Eq. (8-3.2) yields the relation for the radial stress. For rectangular cross section curved beams subjected to shear loading (Fig. 8-2.3*b*), a comparison of the resulting approximate solution with the elasticity solution indicates that the approximate solution is conservative. Furthermore, for such beams it remains conservative to within 6 percent for values of $R/h > 1.0$ even if the term involving N in Eq. (8-3.3) is discarded. Consequently, if we retain only the moment term in Eq. (8-3.3), the expression for the radial stress may be approximated by the formula,

$$\sigma_{rr} = \frac{AA'_m - A'A_m}{trA(RA_m - A)} M_x \tag{8-3.5}$$

to within 6 percent of the elasticity solution for rectangular cross section curved beams subjected to shear loading (Fig. 8-2.3*b*).

EXAMPLE 8-3.1
Radial Stress in T-Section

The curved beam in Fig. E8-3.1 is subjected to a load $P = 120$ kN. The dimensions of section BC are also shown. Determine the circumferential stress at B and the radial stress at the junction of the flange and web at section BC.

SOLUTION

The magnitudes of A, A_m, and R are given by Eqs. (8-2.12), (8-2.13), and (8-2.14), respectively. They are

$$A = 48(120) + 120(24) = 8640 \text{ mm}^2$$

$$R = \frac{48(120)(96) + 120(24)(180)}{8640} = 124.0 \text{ mm}$$

$$A_m = 120 \ln \frac{120}{72} + 24 \ln \frac{240}{120} = 77.93 \text{ mm}$$

The circumferential stress is given by Eq. (8-2.11). It is

$$\sigma_{\theta\theta B} = \frac{120{,}000}{8640} + \frac{364.0(120{,}000)[8640 - 72(77.93)]}{8640(72)[124.0(77.93) - 8640]}$$

$$= 13.9 + 207.8 = 221.7 \text{ MPa}$$

The radial stress at the junction of the flange and web is given by Eq.

(8-3.5), with $r = 120$ mm and $t = 24$ mm. Magnitudes of A' and A'_m are

$$A' = 48(120) = 5760 \text{ m}^2$$

$$A'_m = 120 \ln \frac{120}{72} = 61.30 \text{ mm}$$

Substitution of these values in Eq. (8-3.5) gives

$$\sigma_{rr} = \frac{364.0(120{,}000)[8640(61.30) - 5760(77.93)]}{24(120)(8640)[124.0(77.93) - 8640]} = 138.5 \text{ MPa}$$

Hence, the magnitude of this radial stress is appreciably less than the maximum circumferential stress ($|\sigma_{\theta\theta B}| > |\sigma_{\theta\theta C}|$) and may not be of

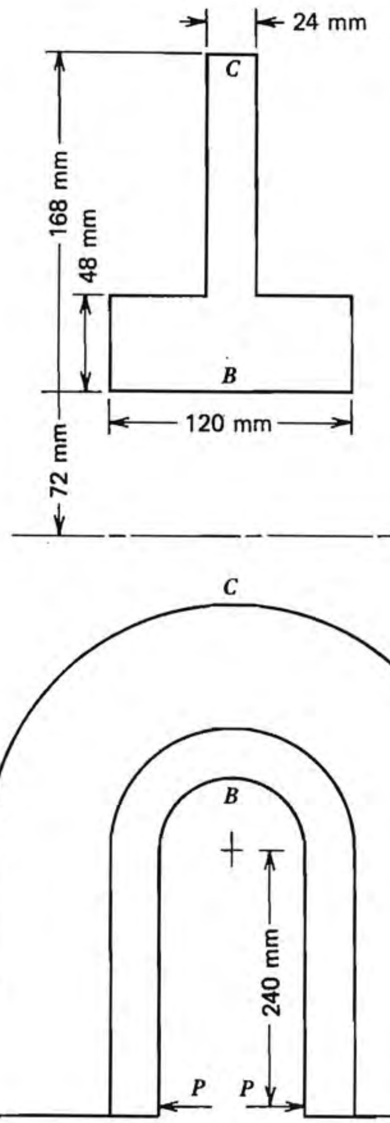


Fig. E8-3.1

concern for the design engineer. However, in the solution of this problem, the effect of the stress concentration at the fillet joining the flange to the web has not been considered. This stress concentration increases the magnitude of the radial stress at the junction of the flange and web. However, the increase in stress is localized. Hence, it is not significant for curved beams made of ductile metal and subjected to static loads. However, for curved beams made of brittle materials or for curved beams of ductile material subjected to repeated loads, the localized stresses are significant. The effect of stress concentrations at fillets are considered in Chapter 12.

PROBLEM SET 8-3

1. For the curved beam in Problem 8-2.5, determine the radial stress in terms of the moment M_x if the thickness of the web is 10 mm.

2. In Fig. P8-3.2 is shown a cast iron frame with a U-shaped cross section. The ultimate tensile strength of the case iron is $\sigma_u = 320$ MPa. (a) Determine the maximum value of P based on a factor of safety $SF = 4.00$ which is based on the ultimate strength. (b) Neglecting the effect of stress concentrations for the fillet at the junction of the web and flange, determine the maximum radial stress when this load is applied. (c) Is the maximum radial stress less than the maximum circumferential stress?

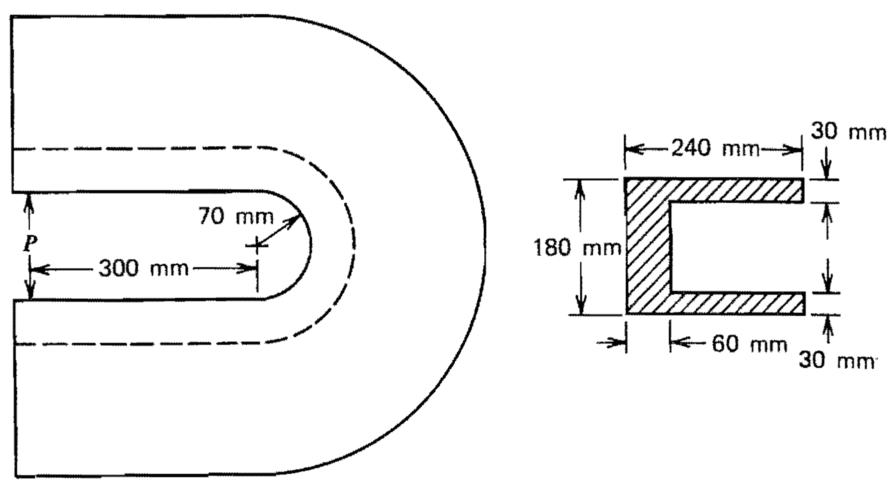


Fig. P8-3.2

Ans. (a) $P = 110.8$ kN (b) $\sigma_{rr} = 42.4$ MPa (c) Yes

8-4 CORRECTION OF CIRCUMFERENTIAL STRESSES IN CURVED BEAMS HAVING I, T, OR SIMILAR CROSS SECTIONS

If the curved beam formula is used to calculate circumferential stresses in curved beams having thin flanges, the computed stresses are considerably in error and the error is nonconservative. The error arises because the radial forces developed in the curved beam causes the outer portion of the flanges to deflect radially, thereby distorting the cross section of the curved beam. The resulting effect is to decrease the stiffness of the curved beam, to decrease the circumferential stresses in the outer portion of the flanges, and to increase the circumferential stresses in the inner portion of the flanges.

Consider a short length of a thin flange I-section curved beam included between faces BC and FH which form an infinitesimal angle $d\theta$ as indicated in Fig. 8-4.1a. If the curved beam is subjected to a positive moment M_x the circumferential stress distribution results in a tensile force T acting on the inner flange and a compressive force C acting on the outer flange, as shown. The components of these forces in the radial direction are $T d\theta$ and $C d\theta$. If the cross section of the curved beam did not distort, these forces would be uniformly distributed along each flange, as indicated in Fig. 8-4.1b. However, the two portions of the tension and compression flanges act as cantilever beams fixed at the web. The resulting bending due to cantilever beam action causes the flanges to distort, as indicated in Fig. 8-4.1c.

The effect of the distortion of the cross section on the circumferential stresses in the curved beam can be determined by examining the portion of the curved beam $ABCD$ in Fig. 8-4.1d. Sections AC and BD are separated by angle θ in the unloaded beam. When the curved beam is subjected to a positive moment, the center of curvature moves from 0 to 0*, section AC moves to A^*C^*, section BD moves to B^*D^*, and the included angle becomes θ^*. If the cross section does not distort, the inner tension flange AB elongates to length A^*B^*. Since the outer portion of the inner flange moves radially inward relative to the undistorted position (Fig. 8-4.1c), the circumferential elongation of the outer portion of the inner flange is less than that indicated in Fig. 8-4.1d. *Therefore,* $\sigma_{\theta\theta}$ *in the outer portion of the inner flange is less than that calculated using the curved beam formula.* In order to satisfy equilibrium, it is necessary that $\sigma_{\theta\theta}$ for the portion of the flange near the web be greater than that calculated using the curved beam formula. Now consider the outer compression flange. As indicated in Fig. 8-4.1d, the outer flange shortens from CD to C^*D^* if the cross section does not distort. Because of the distortion (Fig. 8-4.1c), the outer portion of the compressive flange moves radially outward requiring less compressive contraction. *Therefore, the magnitude*

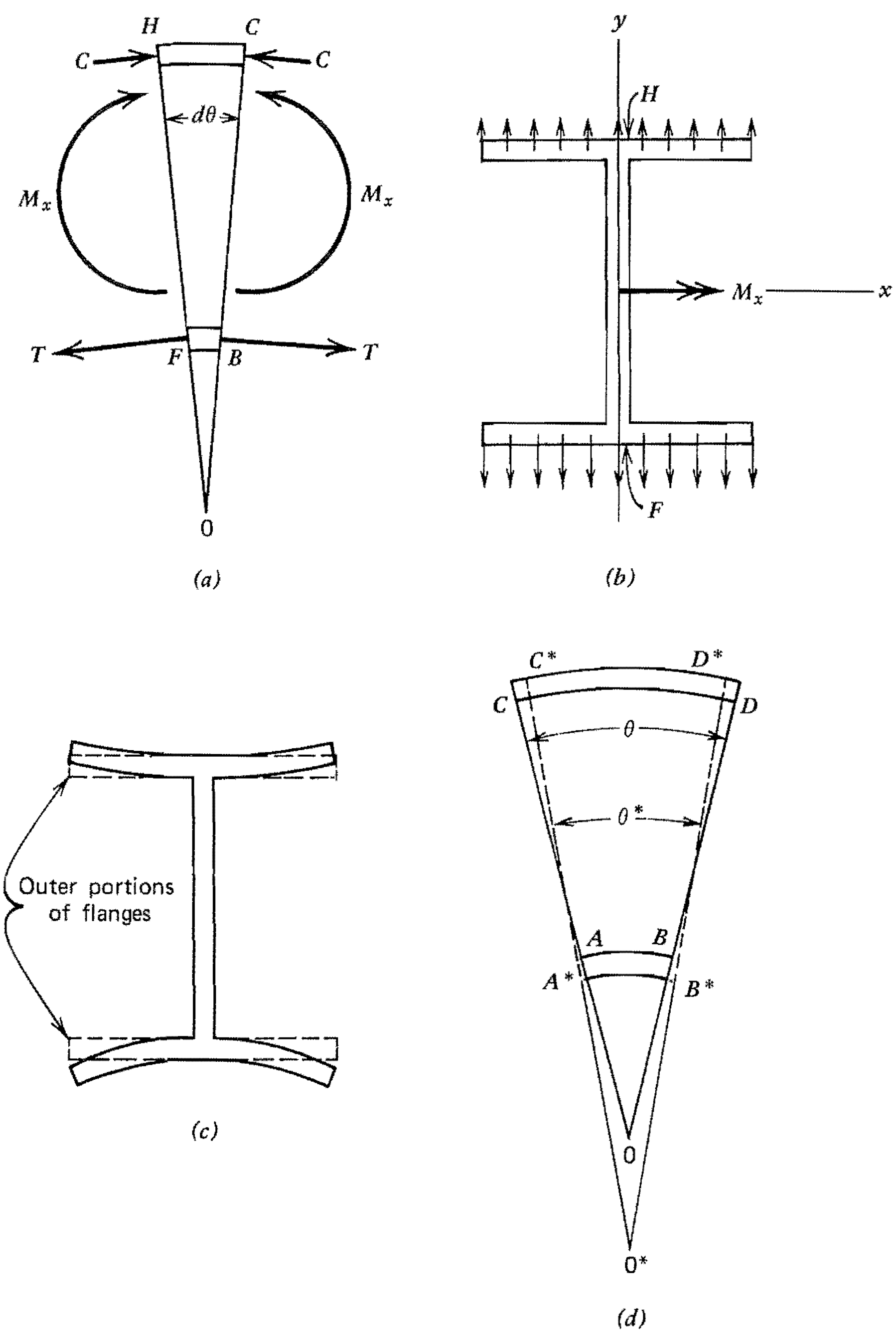


Fig. 8-4.1/Distortion of cross section of an I-section curved beam.

of $\sigma_{\theta\theta}$ in the outer portion of the compression outer flange is less than that calculated by the curved beam formula, and the magnitude of $\sigma_{\theta\theta}$ in the portion of the compression flange near the web is larger than that calculated by the curved beam formula.

The resulting circumferential stress distribution is indicated in Fig. 8-4.2. Since the curved beam formula assumes that the circumferential stress is independent of x (Fig. 8-2.1), corrections are required if the formula is to be used in the design of curved beams having I or T cross sections and similar cross sections. There are two approaches (approxi-

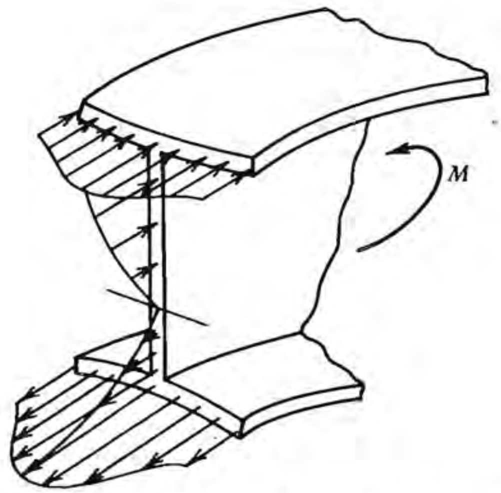


Fig. 8-4.2/Stresses in I-section of curved beam.

mations) that can be employed in the design of these curved beams. One approach is to prevent the radial distortion of the cross section by welding radial stiffeners to the curved beams. If distortion of the cross section is prevented, the use of the curved beam formula is appropriate. A second approach, suggested by H. Bleich,[2] is discussed below.

Bleich's Correction Factors/Bleich reasoned that the actual maximum circumferential stresses in the tension and compression flanges for the I-section curved beam (Fig. 8-4.3*a*) may be calculated by the curved beam formula applied to an I-section curved beam with *reduced flange widths*, as indicated in Fig. 8-4.3*b*. By Bleich's method, if the same bending moment is applied to the two cross sections in Fig. 8-4.3, the computed maximum circumferential tension and compression stresses for the cross section shown in Fig. 8-4.3*b*, with no distortion, is equal to the actual maximum circumferential tension and compression stresses for the cross section in Fig. 8-4.3*a*, with distortion.

The approximate solution proposed by Bleich gives the results presented in tabular form in Table 8-4.1. In order to use the table, the ratio $b_p^2/\bar{r}t_f$ must be calculated where

b_p = projecting width of flange (see Fig. 8-4.3*a*)

$\bar{r}$ = radius of curvature to the center of flange

t_f = thickness of flange

The reduced width b_p' of the projecting part of each flange (Fig. 8-4.3*b*) is given by the relation

$$b_p' = \alpha b_p \tag{8-4.1}$$

where α is obtained from Table 8-4.1 for the computed value of the ratio

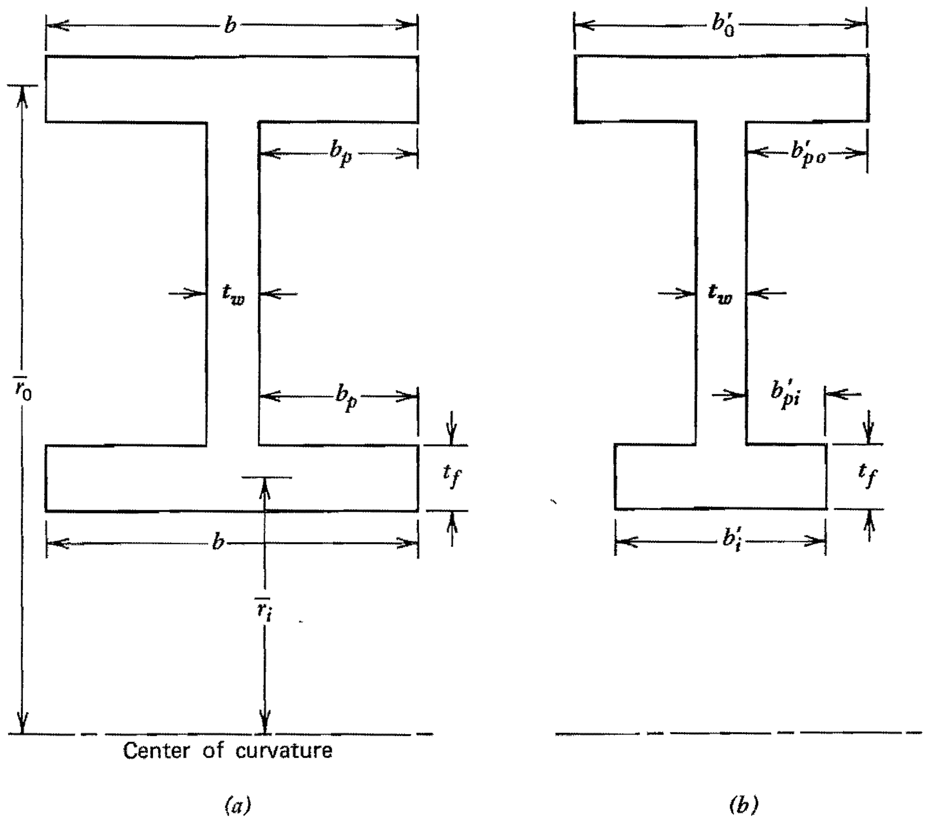


Fig. 8-4.3/Original and modified I-section for a curved beam.

$b_p^2/\bar{r}t_f$. The reduced width of each flange (Fig. 8-4.3b) is given by

$$b' = 2b_p' + t_w \tag{8-4.2}$$

where t_w is the thickness of the web. The curved beam formula (Eq. 8-2.11) when applied to an undistorted cross section corrected by Eq. (8-4.2) predicts the maximum circumferential stress in the actual (distorted) cross section. This maximum stress occurs at the center of the

Table 8-4.1

Table for Calculating the Effective Width and the Lateral Bending Stress of Curved I- or T-Beams

$b_p^2/\bar{r}t$	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
α	0.977	0.950	0.917	0.878	0.838	0.800	0.762	0.726	0.693
β	0.580	0.836	1.056	1.238	1.382	1.495	1.577	1.636	1.677
$b_p^2/\bar{r}t$	1.1	1.2	1.3	1.4	1.5	2.0	3.0	4.0	5.0
α	0.663	0.636	0.611	0.589	0.569	0.495	0.414	0.367	0.334
β	1.703	1.721	1.728	1.732	1.732	1.707	1.671	1.680	1.700

inner flange. It should be noted that the state of stress at this point in the curved beam is not uniaxial. Because of the bending of the flanges (Fig. 8-4.1*c*), an *x*-component of stress σ_{xx} (Fig. 8-2.1) is developed; the sign of σ_{xx} is opposite to that of $\sigma_{\theta\theta(\max)}$. Bleich obtained an approximate solution for σ_{xx} for the inner flange. It is given by the relation

$$\sigma_{xx} = -\beta\bar{\sigma}_{\theta\theta} \tag{8-4.3}$$

where β is obtained from Table 8-4.1 for the computed value of the ratio $b_p^2/\bar{r}t_f$, and where $\bar{\sigma}_{\theta\theta}$ is the magnitude of the circumferential stress at mid-thickness of the inner flange; the value of $\bar{\sigma}_{\theta\theta}$ is calculated based on the corrected cross section.

Although Bleich's analysis was developed for curved beams with relatively thin flanges, the results obtained agree closely with a similar solution obtained by C. G. Anderson[3] for I-beams and box beams in which the analysis was not restricted to thin-flanged sections. Similar analyses of tubular curved beams with circular and rectangular cross sections have been made by T. von Kármán[4] and by S. Timoshenko.[5] An experimental investigation by D. C. Broughton, M. E. Clark, and H. T. Corten[6] showed that another type of correction is needed if the curved beam has extremely thick flanges and thin webs. For such beams each flange tends to rotate about a neutral axis of its own in addition to the rotation about the neutral axis of the curved beam cross section as a whole. Curved beams for which the circumferential stresses are appreciably increased by this action probably fail by excessive radial stresses.

Note: The radial stress can be calculated using either the original or the modified cross section.

EXAMPLE 8-4.1

Bleich Correction Factors for T-Section

A T-section curved beam has the dimensions indicated in Fig. E8-4.1*a* and is subjected to pure bending. The curved beam is made of a steel having a yield stress $Y = 280$ MPa. (a) Determine the magnitude of the moment which indicates yielding in the curved beam if Bleich's correction factors are not used. (b) Use Bleich's correction factors to obtain a modified cross section. Determine the magnitude of the moment that initiates yielding for the modified cross section and compare with the result of Part (a).

SOLUTION

(a) The magnitudes of A, A_m, and R for the original cross section are given by Eq. (8-2.12), (8-2.13) and (8-2.14), respectively, as follows: $A = 4000$ mm^2, $A_m = 44.99$ mm, and $R = 100.0$ mm. By comparison of

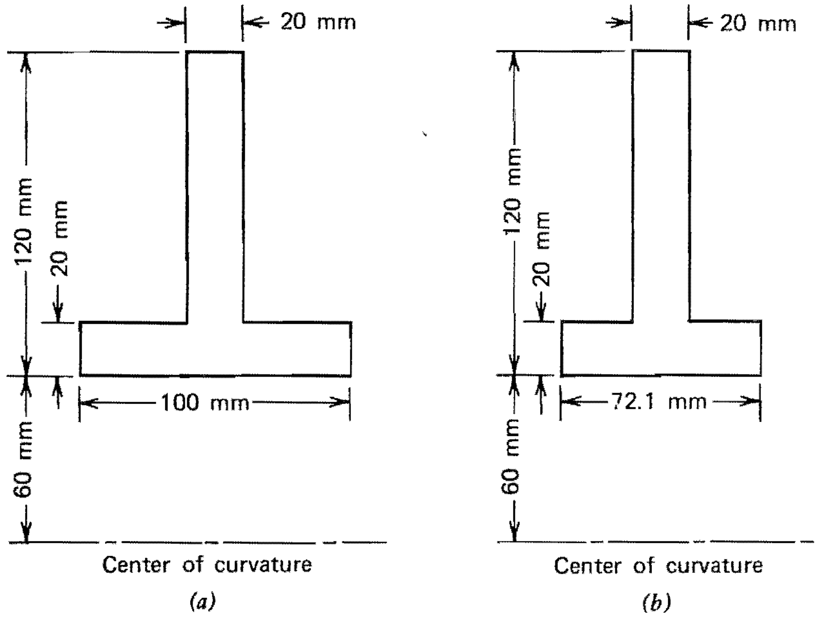


Fig. E8-4.1/(*a*) Original section. (*b*) Modified section.

the stresses at the locations $r = 180$ mm and at $r = 60$ mm, we find that the maximum magnitude of $\sigma_{\theta\theta}$ occurs at the outer radius ($r = 180$ mm). See Eq. (8-2.11). Thus,

$$\sigma_{\theta\theta(\max)} = \left| \frac{M_x[4000 - 180(44.99)]}{4000(180)[100.0(44.99) - 4000]} \right|$$

$$= | -1.141 \times 10^{-5} M_x | \text{ (MPa)}$$

where M_x has the dimensions of N · mm. Since the state of stress is assumed to be uniaxial, the magnitude of M_x to initiate yielding is obtained by setting $\sigma_{\theta\theta} = -Y$. Thus,

$$M_x = \frac{280}{1.141 \times 10^{-5}} = 24{,}540{,}000 \text{ N} \cdot \text{mm} = 24.54 \text{ kN} \cdot \text{m}$$

(b) The dimensions of the modified cross section are computed by Bleich's method; hence, $b_p^2/\bar{r}t_f$ must be calculated. It is

$$\frac{b_p^2}{\bar{r}t_f} = \frac{40(40)}{70(20)} = 1.143$$

A linear interpolation in Table 8-4.1 yields $\alpha = 0.651$ and $\beta = 1.711$. Hence, by Eq. (8-4.2), the modified flange width given is $b_p' = \alpha b_p = 0.651(40) = 26.04$ mm and $b' = 2b_p' + t_w = 2(26.04) + 20 = 72.1$ mm (Fig. E8-4.1*b*). For this cross section, by means of Eqs. (8-2.12), (8-2.13), and

(8-2.14), we find

$$A = 72.1(20) + 20(100) = 3442 \text{ mm}^2$$

$$R = \frac{72.1(20)(70) + 20(100)(130)}{3442} = 104.9 \text{ mm}$$

$$A_m = 72.1 \ln \frac{80}{60} + 20 \ln \frac{180}{80} = 36.96 \text{ mm}$$

Now by means of Eq. (8-2.11), we find that the maximum magnitude of $\sigma_{\theta\theta}$ occurs at the inner radius of the modified cross section. Thus, with $r = 60$ mm, Eq. (8-2.11) yields

$$\sigma_{\theta\theta(\max)} = \frac{M_x[3442 - 60(36.96)]}{3442(60)[104.9(36.96) - 3442]} = 1.363 \times 10^{-5} M_x \text{ (MPa)}$$

The magnitude of M_x that causes yielding can be calculated by means of either the maximum shearing stress criterion of failure or the maximum octahedral shearing stress criterion of failure. If the maximum shearing stress criterion is used, the minimum principal stress must be computed. The minimum principal stress is σ_{xx}. Hence, by Eq. (8-4.3), we find

$$\bar{\sigma}_{\theta\theta} = \frac{M_x[3442 - 70(36.96)]}{3442(70)[104.9(36.96) - 3442]} = 8.15 \times 10^{-6} M_x \text{ (MPa)}$$

$$\sigma_{xx} = -\beta\bar{\sigma}_{\theta\theta} = -1.711(8.15 \times 10^{-6} M_x) = -1.394 \times 10^{-5} M_x \text{ (MPa)}$$

and

$$\tau_{\max} = \frac{\sigma_{\max} - \sigma_{\min}}{2} = \frac{Y}{2} = \frac{\sigma_{\theta\theta(\max)} - \sigma_{xx}}{2}$$

$$M_x = 10{,}140{,}000 \text{ N} \cdot \text{mm} = 10.14 \text{ kN} \cdot \text{m}$$

A comparison of the moment M_x as determined in Parts (a) and (b) above indicates that the computed M_x required to initiate yielding is reduced by 58.8 percent because of the distortion of the cross section. Since the yielding is highly localized, its effect is not of concern unless the curved beam is subjected to fatigue loading (see Art. 3-5). If the second principal stress σ_{xx} is neglected, the moment M_x is reduced by 16.5 percent because of the distortion of the cross section. The distortion is reduced if the flange thickness is increased.

PROBLEM SET 8-4

1. A T-section curved beam has the cross section shown in Fig. P8-4.1. The center of curvature lies 40 mm from the flange. If the curved beam is subjected to a positive bending moment $M_x = 2.50$ kN · m, determine the stresses at the inner and outer radii. Use Bleich's

correction factors. What is the maximum shearing stress in the curved beam?

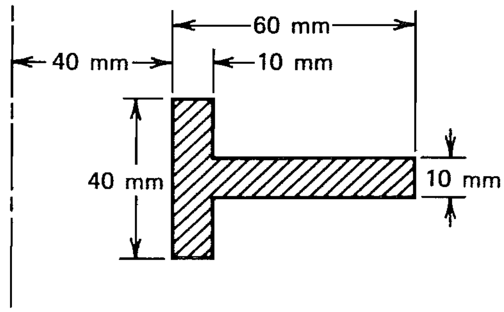


Fig. P8-4.1

2. Determine the radial stress at the junction of the web and the flange for the curved beam in Problem 1. Neglect stress concentrations.

Ans. $\sigma_{rr} = 118.1$ MPa

3. A load $P = 12.0$ kN is applied to the clamp shown in Fig. P8-4.3. Determine the circumferential stresses at points B and C assuming that the curved beam formula is valid at that section.

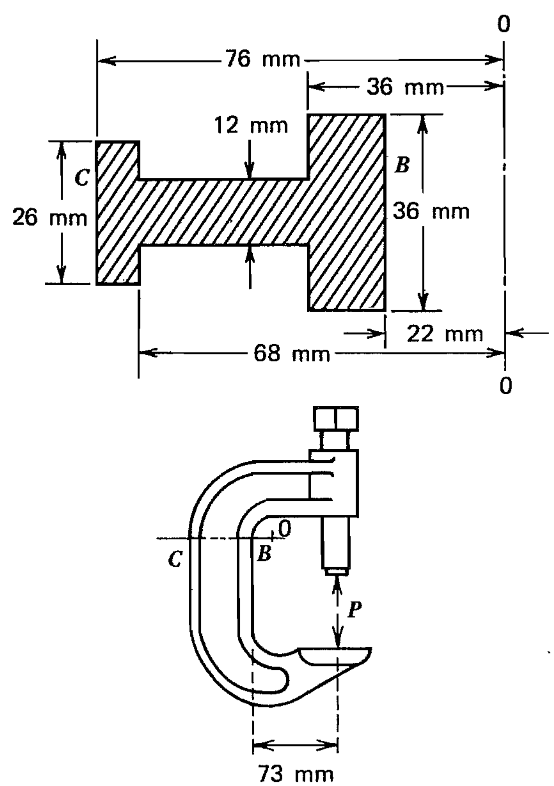


Fig. P8-4.3

4. Determine the radial stress at the junction of the web and inner flange of the curved beam portion of the clamp in Problem 3. Neglect stress concentrations.

   Ans. $\sigma_{rr} = 69.7$ MPa

8-5 DEFLECTIONS OF CURVED BEAMS

A convenient method of determining the deflections of a linearly elastic curved beam is by the use of Castigliano's theorem (Chapter 4). For example, the deflections of the free end of the curved beam in Fig. 8-2.1*a* are given by the relations

$$\delta_{P_1} = \frac{\partial U}{\partial P_1} \tag{8-5.1}$$

$$\theta = \frac{\partial U}{\partial M_0} \tag{8-5.2}$$

where δ_{P_1} is the component of the deflection of the free end of the curved beam in the direction of load P_1, θ is the angle of rotation of the free end of the curved beam in the direction of M_0, and U is the total elastic strain energy in the curved beam. The total strain energy U (see Eq. 4-3.3) is equal to the integral of the strain energy density U_0 over the volume of the curved beam (see Eqs. 2-4.10 and 4-3.4).

Consider the strain energy density U_0 for a curved beam (Fig. 8-2.1). Because of the symmetry of loading relative to the (y, z)-plane, $\sigma_{xy} = \sigma_{xz} = 0$, and since the effect of the transverse normal stress σ_{xx} (Fig. 8-2.1*b*) is ordinarily neglected, the formula for the strain energy density U_0 reduces to the form

$$U_0 = \frac{1}{2E}\sigma_{\theta\theta}^2 + \frac{1}{2E}\sigma_{rr}^2 - \frac{\nu}{E}\sigma_{rr}\sigma_{\theta\theta} + \frac{1}{2G}\sigma_{r\theta}^2$$

where the radial normal stress σ_{rr}, the circumferential normal stress $\sigma_{\theta\theta}$, and the shearing stress $\sigma_{r\theta}$ are, relative to the (x, y, z)-axes of Fig. 8-2.1*b*, $\sigma_{rr} = \sigma_{yy}$, $\sigma_{\theta\theta} = \sigma_{zz}$, and $\sigma_{r\theta} = \sigma_{yz}$. In addition, the effect of σ_{rr} is often small for curved beams of practical dimensions. Hence, the effect of σ_{rr} is often discarded from the expression for U_0. Then,

$$U_0 = \frac{1}{2E}\sigma_{\theta\theta}^2 + \frac{1}{2G}\sigma_{r\theta}^2$$

The stress components $\sigma_{\theta\theta}$ and $\sigma_{r\theta}$, respectively, contribute to the strain

energies U_N and U_S because of the normal traction N and the shear V (Fig. 8-2.1b). In addition, $\sigma_{\theta\theta}$ contributes to the bending strain energy U_M, as well as to a strain energy U_{MN} because of a coupling effect between the moment M and the traction N, as we shall see in the derivation below.

If curved beams have a small length to depth ratio, the curved beam formula should be used to calculate the circumferential stress $\sigma_{\theta\theta}$. Ordinarily it is sufficiently accurate to approximate the strain energies U_S and U_N that are due to shear V and traction N, respectively, by the formulas for straight beams (see Art. 4-3). However, the strain energy U_M due to bending must be modified. To compute the strain energy due to bending, consider the curved beam shown in Fig. 8-2.1b. Since the strain energy increment dU for a linearly elastic material undergoing small displacement is independent of the order in which loads are applied, let the shear load V and normal load N be applied first. Next, let the moment be increased from zero to M_x. The strain energy increment due to bending is

$$dU_M = \tfrac{1}{2}M_x\Delta(d\theta) = \tfrac{1}{2}M_x\omega\, d\theta \tag{8-5.3}$$

where $\Delta(d\theta)$, the change in $d\theta$, and $\omega = \Delta(d\theta)/d\theta$ are due to M_x alone. Hence, ω is determined from Eq. (8-2.10) with $N=0$. Consequently, Eqs. (8-5.3) and Eq. (8-2.10) yield (with $N=0$)

$$dU_M = \frac{A_m M_x^2}{2A(RA_m - A)E}\, d\theta \tag{8-5.4}$$

During the application of M_x, additional work is done by N because the centroidal (middle) surface, (Fig. 8-2.1b) is stretched an amount $d\bar{e}_{\theta\theta}$. Let the corresponding strain energy increment due to the stretching of the middle surface be denoted by dU_{MN}. This strain energy increment dU_{MN} is equal to the work done by N as it moves through the distance $d\bar{e}_{\theta\theta}$. Thus,

$$dU_{MN} = N\, d\bar{e}_{\theta\theta} = N\bar{\epsilon}_{\theta\theta} R\, d\theta \tag{8-5.5}$$

where $d\bar{e}_{\theta\theta}$ and $\bar{\epsilon}_{\theta\theta}$ refer to the elongation and strain of the centroidal axis, respectively. The strain $\bar{\epsilon}_{\theta\theta}$ is given by Eq. (8-2.3) with $r=R$. Thus, Eq. (8-2.3) (with $r=R$) and Eqs. (8-5.5), (8-2.9), and (8-2.10) (with $N=0$) yield the strain energy increment dU_{MN} due to coupling of the moment M_x and traction N.

$$dU_{MN} = \frac{N}{E}\left[\frac{M_x}{RA_m - A} - R\frac{A_m M_x}{A(RA_m - A)}\right] d\theta = -\frac{M_x N}{EA}\, d\theta \tag{8-5.6}$$

By Eqs. (4-3.6), (4-3.11), (8-5.4), and (8-5.6), the total strain energy U for the curved beam is obtained in the form

$$U = \int \frac{kV^2R}{2AG}\,d\theta + \int \frac{N^2R}{2AE}\,d\theta + \int \frac{A_m M_x^2}{2A(RA_m - A)E}\,d\theta - \int \frac{M_x N}{EA}\,d\theta \tag{8-5.7}$$

Equation (8-5.7) is an approximation, since it is based on the assumptions that plane sections remain plane and that the effect of the radial stress σ_{rr} on U is negligible. It might be expected that the radial stress increases the strain energy. Hence, Eq. (8-5.7) yields a low estimate of the actual strain energy. However, if M_x and N have the same sign, the coupling energy U_{MN}, the last term in Eq. (8-5.7), is negative. Ordinarily, U_{MN} is small and, in many cases, it is negative. Hence, we recommend that U_{MN}, the coupling strain energy be discarded from Eq. (8-5.7) when it is negative. The discarding of U_{MN} from Eq. (8-5.7) raises the estimate of the actual strain energy when U_{MN} is negative, and compensates to some degree for the lower estimate due to discarding σ_{rr}.

The deflection δ_{elast} of rectangular cross section curved beams has been given by Timoshenko[1] for the two types of loading shown in Fig. 8-2.3. The ratio of the deflection δ_U given by Castigliano's theorem and the deflection δ_{elast} is presented in Table 8-5.1 for several values of R/h. The shear coefficient k (see Eqs. 4-3.11 and Table 4-3.1) was taken to be 1.5 for the rectangular section, and Poisson's ratio ν was assumed to be 0.30.

Note: The deflection of curved beams is much less influenced by the curvature of the curved beam than is the circumferential stress $\sigma_{\theta\theta}$. If

Table 8-5.1

Ratios of Deflections in Rectangular Section Curved Beams as Computed by Elasticity Theory and by Approximate Strain Energy Solution

	Neglecting U_{MN}		*Including U_{MN}*	
	Pure Bending	*Shear Loading*	*Pure Bending*	*Shear Loading*
$\left(\frac{R}{h}\right)$	$\left(\frac{\delta_U}{\delta_{elast}}\right)$	$\left(\frac{\delta_U}{\delta_{elast}}\right)$	$\left(\frac{\delta_U}{\delta_{elast}}\right)$	$\left(\frac{\delta_U}{\delta_{elast}}\right)$
0.65	0.923	1.563	0.697	1.215
0.75	0.974	1.381	0.807	1.123
1.0	1.004	1.197	0.914	1.048
1.5	1.006	1.085	0.968	1.016
2.0	1.004	1.048	0.983	1.008
3.0	1.002	1.021	0.993	1.003
5.0	1.000	1.007	0.997	1.001

R/h is greater than 2.0, the strain energy due to bending can be approximated by that for a straight beam. Thus, for $R/h > 2.0$, for computing deflections the third and fourth terms on the right-hand side of Eq. (8-5.7) may be replaced by

$$U_M = \int \frac{M_x^2}{2EI_x} R\,d\theta \tag{8-5.8}$$

In particular, we note that the deflection of a rectangular cross section curved beam with $R/h = 2.0$ is 7.7 percent greater when the curved beam is assumed to be straight than when it is assumed to be curved.

Deflections of Curved Beam. Cross Sections in the Form of an I, T, etc./ As discussed in Art. 8-4, the cross sections of curved beams in the form of an I, T, etc., undergo distortion when loaded. One effect of the distortion is to decrease the stiffness of the curved beam. As a result, deflections calculated on the basis of the undistorted cross section are less than the actual deflections. Therefore, the deflection calculations should be based upon modified cross sections determined by Bleich's correction factors (Table 8-4.1). The strain energy terms U_N and U_M for the curved beams should also be calculated using the modified cross section. We recommend that the strain energy U_S be calculated with $k = 1.0$, and with the cross-sectional area A replaced by the area of the web $A_w = th$, where t is the thickness of the web and h is the curved beam depth. Also, as a working rule, we recommend that the coupling energy U_{MN} be neglected if it is negative, and that it be doubled if it is positive.

EXAMPLE 8-5.1

Deformations in a Curved Beam Subjected to Pure Bending

The curved beam in Fig. E8-5.1 is made of an aluminum alloy ($E = 72.0$ GPa), has a rectangular cross section with a depth of 60 mm, and is

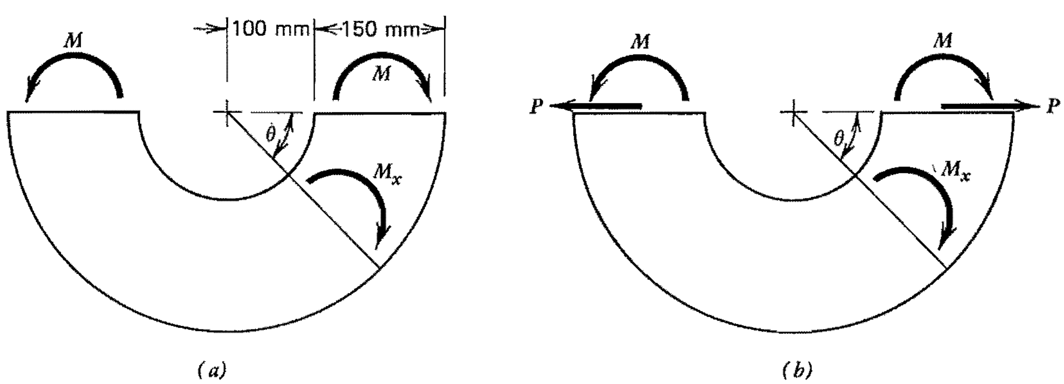


Fig. E8-5.1

subjected to a pure bending moment $M = 24.0$ kN · m. (a) Determine the angle change between the two horizontal faces where M is applied. (b) Determine the relative displacement of the centroids of the horizontal faces of the curved beam.

SOLUTION

Required values for A, A_m, and R for the curved beam are calculated using equations in Row a of Table 8-2.2.

$$A = 60(150) = 9000 \text{ mm}^2$$

$$A_m = 60 \ln \frac{250}{100} = 54.98 \text{ mm}$$

$$R = 100 + 75 = 175 \text{ mm}$$

(a) The angle change between the two faces where M is applied is given by Eq. (8-5.2). As indicated in Fig. E8-5.1a, the magnitude of M_x at any angle θ is $M_x = M$. Thus, by Eq. (8-5.2), we obtain

$$\theta = \frac{\partial U}{\partial M} = \int_0^{\pi} \frac{A_m M_x}{A(RA_m - A)E}(1)\, d\theta$$

$$= \frac{54.98(24{,}000{,}000)\pi}{9000[175(54.98) - 9000](72{,}000)}$$

$$= 0.01029 \text{ rad}$$

(b) In order to determine the deflection of the curved beam, a load P must be applied as indicated in Fig. E8-5.1b. In this case $M_x = M + PR \sin\theta$ and $\partial U/\partial P = R\sin\theta$. Then the deflection is given by Eq. (8-5.1), in which the integral is evaluated with $P = 0$. Thus, the relative displacement is given by the relation

$$\delta_P = \frac{\partial U}{\partial P} = \int_0^{\pi} \left. \frac{A_m M_x}{A(RA_m - A)E} \right|_{P=0} (R\sin\theta)\, d\theta$$

or

$$\delta_P = \frac{54.98(24{,}000{,}000)(175)(2)}{9000[175(54.98) - 9000](72{,}000)} = 1.147 \text{ mm}$$

EXAMPLE 8-5.2

Deflections in a Press

The press (Fig. E8-5.2a) has the cross section shown in Fig. E8-5.2b. It is subjected to a load $P = 11.2$ kN. The press is made of a steel with $E = 200$ GPa, and $\nu = 0.30$. Determine the separation of the jaws of the press due to the load.

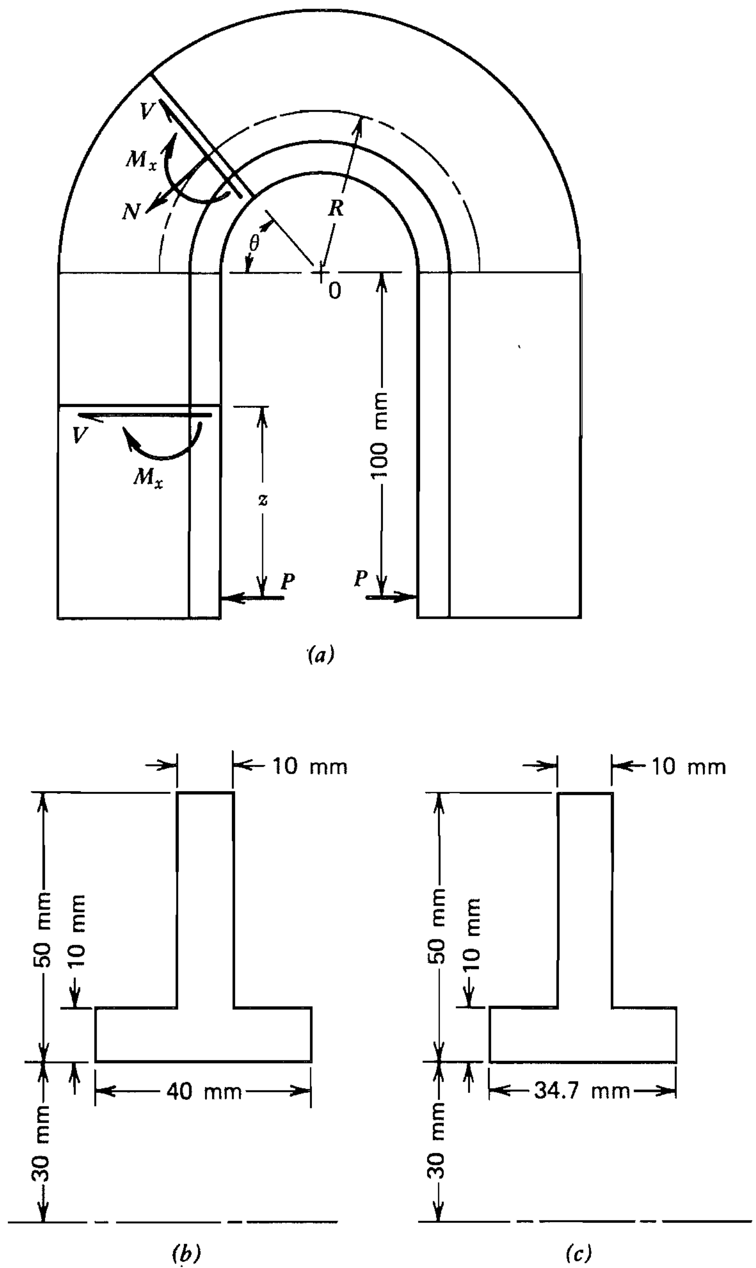


Fig. E8-5.2/(b) Original section. (*c*) Modified section.

SOLUTION

The press is made up of two straight members and a curved member. We compute the strain energies due to bending and shear in the straight beams, without modification of the cross sections. The moment of inertia of the cross section is $I_x = 181.7 \times 10^3$ mm^4. We choose the origin of the coordinate axes at load P, with z measured from P toward the curved beam. Then the applied loads shear V and moment M_x at a section in the

straight beam is

$$V = P$$
$$M_x = Pz$$

In the curved beam portion of the press, we employ Bleich's correction factor to obtain a modified cross section. With the dimensions in Fig. E8-5.2*b*, we find

$$\frac{b_p^2}{\bar{r}t_f} = \frac{15^2}{35(10)} = 0.643$$

A linear interpolation in Table 8-4.1 yields the result $\alpha = 0.822$. The modified cross section is shown in Fig. E8-5.2*c*. Equations (8-2.12), (8-2.13), and (8-2.14) give

$$A = 34.7(10) + 10(40) = 747 \text{ mm}^2$$
$$R = \frac{34.7(10)(35) + 10(40)(60)}{747} = 48.4 \text{ mm}$$
$$A_m = 10 \ln\frac{80}{40} + 34.7 \ln\frac{40}{30} = 16.9 \text{ mm}$$

With θ defined as indicated in Fig. E8-5.2*a*, the applied loads shear V, normal load N, and moment M_x for the curved beam are

$$V = P\cos\theta$$
$$N = P\sin\theta$$
$$M_x = P(100 + R\sin\theta)$$

Summing the strain energy terms for the two straight beams and the curved beam and taking the derivative with respect to P (Eq. 8-5.1), we compute the increase in distance δ_P between the load points as

$$\delta_P = 2\int_0^{100} \frac{P}{A_w G}\, dz + 2\int_0^{100} \frac{Pz^2}{EI_x}\, dz + \int_0^{\pi} \frac{P\cos^2\theta}{A_w G} R\, d\theta + \int_0^{\pi} \frac{P\sin^2\theta}{AE} R\, d\theta$$
$$+ \int_0^{\pi} \frac{P(100 + R\sin\theta)^2 A_m}{A(RA_m - A)E}\, d\theta$$

The shearing modulus is $G = E/[2(1+\nu)] = 76{,}900$ MPa. Hence,

$$\delta_P = \frac{2(11{,}200)(100)}{76{,}900(500)} + \frac{2(11{,}200)(100)^3}{3(200{,}000)(181{,}700)}$$
$$+ \frac{11{,}200(48.4)\pi}{500(76{,}900)(2)} + \frac{11{,}200(48.4)\pi}{747(200{,}000)(2)}$$
$$+ \frac{16.9(11{,}200)}{747[48.4(16.9) - 747](200{,}000)}\left[(100)^2\pi + (48.4)^2\frac{\pi}{2} + 2(100)(48.4)(2)\right]$$

or

$$\delta_P = 0.058 + 0.205 + 0.022 + 0.006 + 0.972 = 1.263 \text{ mm}$$

PROBLEM SET 8-5

1. The curved beam in Fig. P8-5.1 is made of a steel ($E = 200$ GPa) that has a yield point stress $Y = 420$ MPa. Determine the magnitude of the bending moment $M_x = M_Y$ required to initiate yielding in the curved beam, the angle change of the free end, and the horizontal and vertical components of the deflection of the free end.

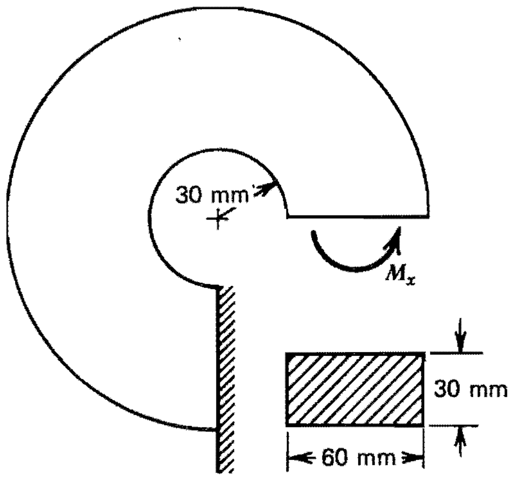


Fig. P8-5.1

2. Determine the deflection of the curved beam in Problem 8-2.3 at the point of load application. The curved beam is made of an aluminum alloy for which $E = 72.0$ GPa and $G = 27.1$ GPa. Let $k = 1.3$.

   Ans. $\delta_P = 0.1629$ mm

3. The triangular cross section curved beam in Problem 8-2.8 is made of steel ($E = 200$ GPa, $G = 77.5$ GPa). Determine the separation of the points of application of the load. Let $k = 1.5$.

4. Determine the deflection across the center of curvature of the cast iron curved beam in Problem 8-3.2 when $P = 126$ kN. $E = 102.0$ GPa and $G = 42.5$ GPa. Let $k = 1.0$ with the area in shear equal to the product of the web thickness and the depth.

   Ans. $\delta_Q = 0.3449$ mm

8.6
STATICALLY INDETERMINATE CURVED BEAMS. CLOSED RING SUBJECTED TO A CONCENTRATED LOAD

Many loaded curved members, such as closed rings and chain links, are statically indeterminate (see Art. 4-5). For such members, equations of

equilibrium are not sufficient to determine all of the loads (V, N, M_x) at a section of the member. The additional relations needed to solve for the loads are obtained using Castigliano's theorem with known boundary conditions on the deformations. Since closed rings are commonly used in engineering, we present the computational procedure for a closed ring.

Consider a closed ring subjected to a central load P (Fig. 8-6.1*a*). From the condition of symmetry, the deformation of each quadrant of the ring is identical. Hence, we need consider only one quadrant. The quadrant (Fig. 8-6.1*b*) may be considered fixed at section FH with a load $P/2$ and moment M_0 at section BC. Because of the symmetry of the ring, as the ring deforms, section BC remains perpendicular to section FH. Therefore, by Castigliano's theorem, we have for the rotation of face BC

$$\phi_{BC} = \frac{\partial U}{\partial M_0} = 0 \tag{8-6.1}$$

The applied loads V, N, and M_x at a section forming angle θ with the face BC are

$$\begin{aligned} V &= \frac{P}{2}\sin\theta \\ N &= \frac{P}{2}\cos\theta \\ M_x &= M_0 - \frac{PR}{2}(1-\cos\theta) \end{aligned} \tag{8-6.2}$$

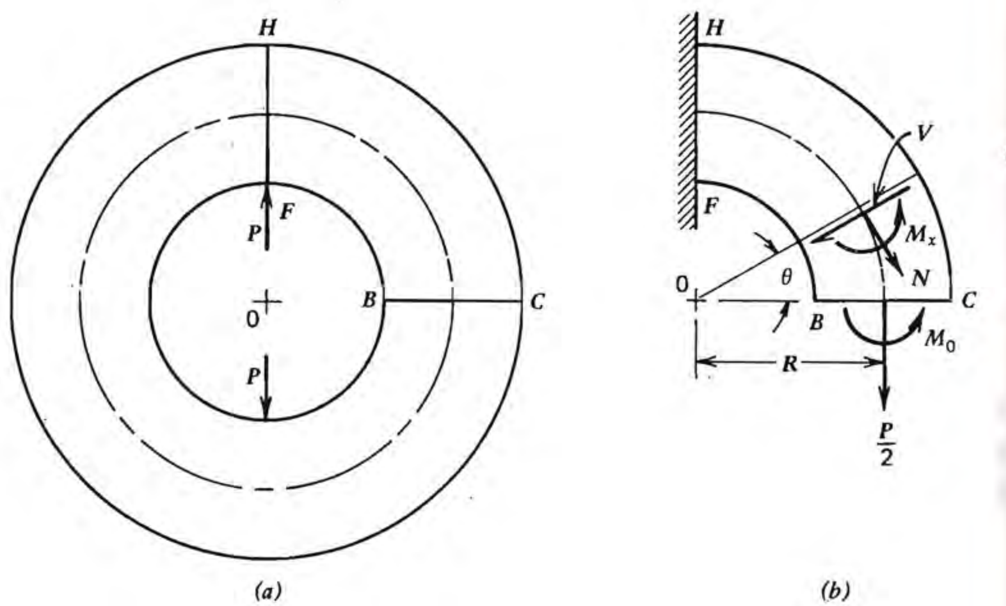


Fig. 8-6.1 / Closed ring.

Substituting Eq. (8-5.7) and Eq. (8-6.2) into Eq. (8-6.1), we find

$$0=\int_0^{\pi/2}\frac{\left[M_0-\dfrac{PR}{2}(1-\cos\theta)\right]A_m}{A(RA_m-A)E}\,d\theta-\int_0^{\pi/2}\frac{\dfrac{P}{2}\cos\theta}{AE}\,d\theta \tag{8-6.3}$$

where U_{MN} has been included. The solution of Eq. (8-6.3) is

$$M_0=\frac{PR}{2}\left(1-\frac{2A}{RA_m\pi}\right) \tag{8-6.4}$$

If R/h is greater than 2.0, we take the bending energy U_M as given by Eq. (8-5.8) and ignore the coupling energy U_{MN}. Then, M_0 is given by the relation

$$M_0=\frac{PR}{2}\left(1-\frac{2}{\pi}\right) \tag{8-6.5}$$

With M_0 known, the loads at every section of the closed ring (Eqs. 8-6.2) are known. The stresses and deformations of the closed ring may be calculated by the methods of Arts. 8-2 to 8-5.

PROBLEM SET 8-6

1. The ring in Fig. P8-6.1 has an inside diameter of 100 mm, an outside diameter of 180 mm, and a circular cross section. The ring is made of a steel having a yield stress $Y = 520$ MPa. Determine the maximum allowable magnitude of P if the ring has been designed with a factor of safety $SF = 1.75$ against initiation of yielding.

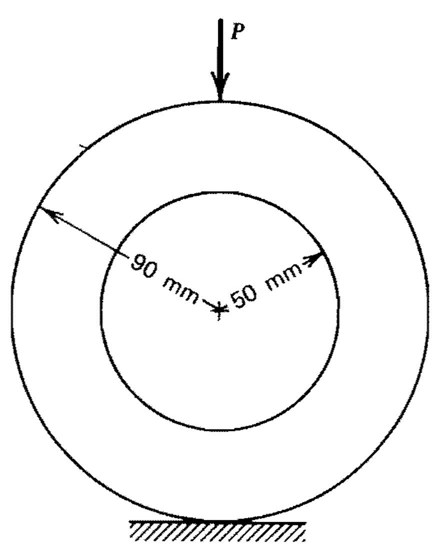


Fig. P8-6.1

2. If $E = 200$ GPa and $G = 77.5$ GPa for the steel in Problem 1, determine the deflection of the ring for a load $P = 60$ kN. Let $k = 1.3$.

   Ans. $\delta_P = 2.088$ mm

3. An aluminum alloy ring has a mean diameter of 600 mm and has a rectangular cross section with 200 mm width and a depth of 300 mm (radial direction). The ring is loaded by diametrically opposite radial loads $P = 4.00$ MN. Determine the maximum tensile and compressive circumferential stresses in the ring.

4. If $E = 72.0$ GPa and $G = 27.1$ GPa for the aluminum alloy ring in Problem 3, determine the separation of the points of application of the loads. Let $k = 1.5$.

   Ans. $\delta_P = 8.742$ mm

5. The link in Fig. P8-6.5 has a circular cross section and is made of a steel having a yield point stress of $Y = 250$ MPa. Determine the magnitude of P that will initiate yielding in the link.

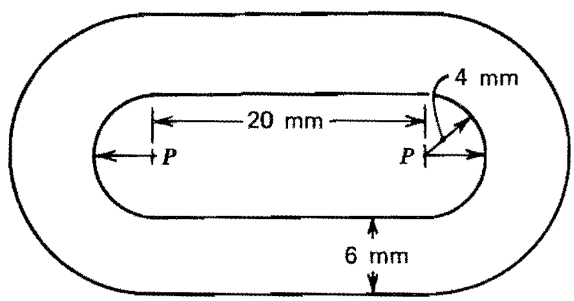


Fig. P8-6.5

8-7
FULLY PLASTIC LOADS FOR CURVED BEAMS

In this article we consider curved beams made of elastic-perfectly plastic materials with yield point stress Y (Fig. 2-6.2*a*). For a curved beam made of elastic-perfectly plastic material, the fully plastic moment M_P under pure bending is the same as that for a straight beam with identical cross section and material. However, because of the nonlinear distribution of the circumferential stress $\sigma_{\theta\theta}$ in a curved beam, the ratio of the fully plastic moment M_P under pure bending to maximum elastic moment M_Y is much greater for a curved beam than for a straight beam with the same cross section.

Most curved beams are subjected to complex loading other than pure bending. The stress distribution for a curved beam at the fully plastic load P_P for a typical loading condition is indicated in Fig. 8-7.1. Since the tension stresses must balance the compression stresses and the load P_P, the part A_T of the cross-sectional area A that has yielded in tension is greater than the part A_C of area A that has yielded in compression. In addition to the unknowns A_T and A_C, a third unknown is P_P, the load at the fully plastic condition. This follows from the fact that R can be calculated, and D is generally specified rather than P_P. The three equations necessary to determine the three unknowns A_T, A_C, and P_P are obtained from the equations of equilibrium and the fact that the sum of A_T and A_C must equal the cross sectional area A, that is,

$$A = A_T + A_C \tag{8-7.1}$$

The equilibrium equations are (Fig. 8-7.1)

$$\sum F_z = 0 = A_T Y - A_C Y - P_P \tag{8-7.2}$$

$$\sum M_x = 0 = P_P D - A_T Y \bar{y}_T - A_C Y \bar{y}_C \tag{8-7.3}$$

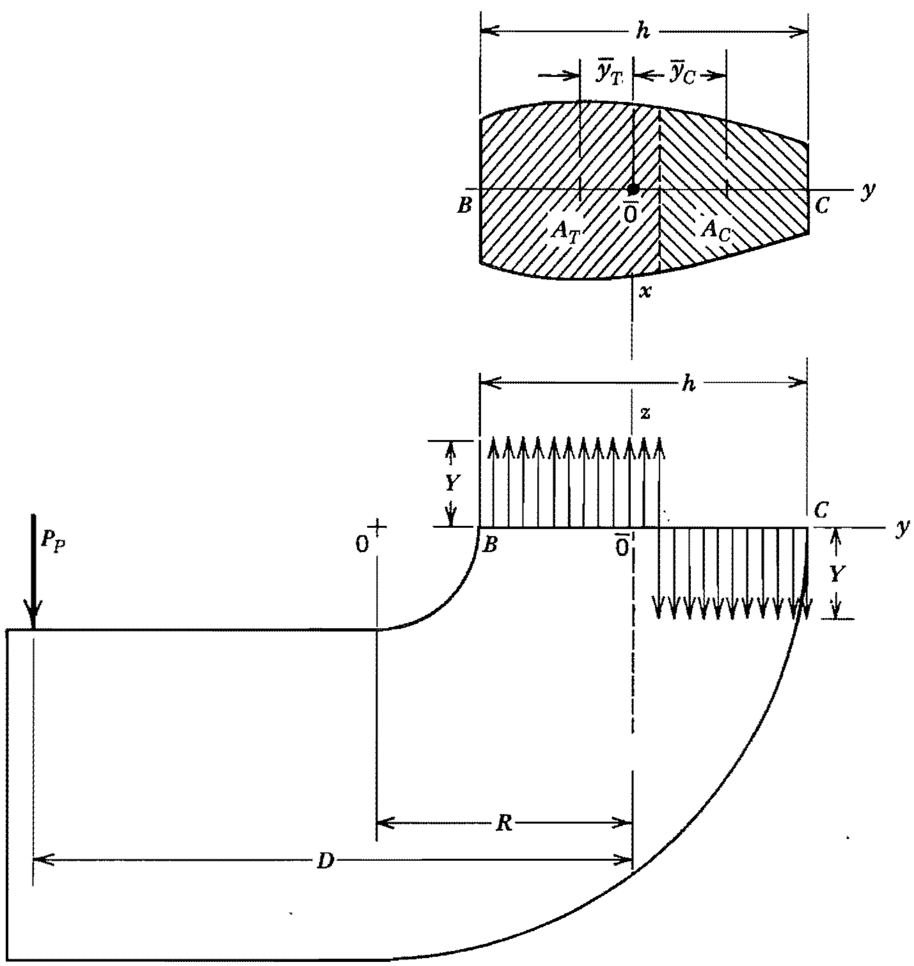


Fig. 8-7.1/Stress distribution for a fully plastic load on a curved beam.

In Eq. (8-7.3), $\bar{y}_T$ and $\bar{y}_C$ locate the centroids of A_T and A_C, respectively, as measured from the centroid of the cross-sectional area of the curved beam (Fig. 8-7.1). Let M be the moment, about the centroid axis x, resulting from the stress distribution on section BC (Fig. 8-7.1). Then,

$$M = P_P D = A_T Y \bar{y}_T + A_C Y \bar{y}_C \tag{8-7.4}$$

The most convenient method of solving Eqs. (8-7.1), (8-7.2), and (8-7.4) for the magnitudes of A_T, A_C, and P_P is often a trial and error procedure, since $\bar{y}_T$ and $\bar{y}_C$ are not known until A_T and A_C are known.[7]

The moment M (Eq. 8-7.4) is generally less than the fully plastic moment M_P for pure bending. It is desirable to know the conditions under which M due to load P_P can be assumed equal to M_P, since for pure bending A_T is equal to A_C, and the calculations are greatly simplified. When the distance D is greater than the depth h of the curved beam, M is approximately equal to M_P for some common sections. For example, for $D = h$, we note that $M = 0.94M_P$ for curved beams with rectangular sections and $M = 0.96M_P$ for curved beams with circular sections. However, for curved beams with T-sections, M may be greater than M_P. Other exceptions are curved beams with I-sections and box sections, for which D should be greater than $2h$ in order for M to be approximately equal to M_P.

Fully Plastic versus Maximum Elastic Loads for Curved Beams / A linearly elastic analysis of a load carrying member is required in order to predict the load-deflection relation of linearly elastic behavior of the member up to the load P_Y that initiates yielding in the member. The fully plastic load is also of interest since it is often considered to be the limiting load that can be applied to the member before the deformations become excessively large.

The fully plastic load P_P for a curved beam is often more than twice the maximum elastic load P_Y. Fracture loads for curved beams that are made of ductile metals and that are subjected to static loading may be 4 to 6 times P_Y. Dimensionless load-deflection experimental data for a uniform rectangular section hook made of a structural steel are shown in Fig. 8-7.2. The deflection is defined as the change in distance ST between points S and T on the hook. The hook did not fracture even for loads such that $P/P_Y > 5$. A computer program written by J. C. McWhorter, H. R. Wetenkamp, and O. M. Sidebottom[7] gave the predicted curve in Fig. 8-7.2. The experimental data agree well with predicted results.

As noted in Fig. 8-7.2, the ratio of P_P to P_Y is 2.44. Furthermore, the load-deflection curve does not level off at the fully plastic load, but continues to rise. This behavior may be attributed to strain-hardening. Because of the steep stress gradient in the hook, the strains in the most strained fibers become so large that the material in the most strained

fibers begins to strain harden before yielding can penetrate to sufficient depth at section BC in the hook to develop the fully plastic load.

The usual practice in predicting the deflection of a structure at the fully plastic load is to calculate the deflection of the structure at the fully plastic load, assuming that the structure behaves in a linearly elastic manner up to the fully plastic load (point Q in Fig. 8-7.2) and multiplying this deflection by the ratio P_P/P_Y (in this case 2.44). In this case, with this procedure (Fig. 8-7.2) the resulting calculated deflection (approximately calculated as $2.44(2.4) = 5.9$) is greater than the measured deflection.

Usually, curved members such as crane hooks and chains are not subjected to a sufficient number of repetitions of peak loads during their life for fatigue failure to occur. Therefore, the working loads for these members are often obtained by application of a factor of safety to the fully plastic loads. It is not uncommon to have the working load as great or greater than the maximum elastic load P_Y.

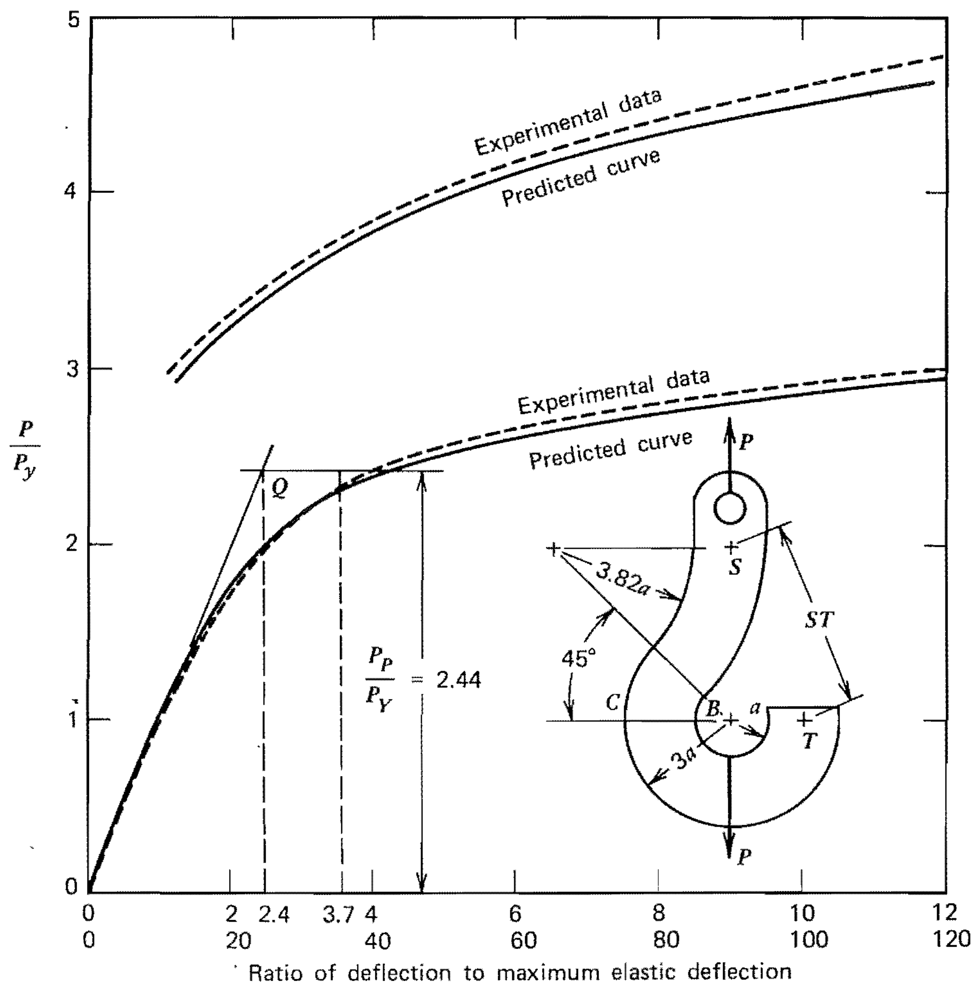


Fig. 8-7.2/Dimensionless load-deflection curves for a uniform rectangular section hook made of structural steel.

PROBLEM SET 8-7

1. Let the curved beam in Fig. 8-7.1 have a rectangular cross section with depth h and width b. Show that the ratio of the bending moment for fully plastic load P_P to the fully plastic moment for pure bending $M_P = Ybh^2/4$ is given by the relation,

$$\frac{M}{M_P} = \frac{4D}{h}\sqrt{1+\frac{4D^2}{h^2}} - \frac{8D^2}{h^2}$$

2. Let the curved beam in Problem 8-2.1 be made of a steel that has a flat top stress-strain diagram at the yield point stress $Y = 430$ MPa. From the answer to Problem 8-2.1 the load that initiates yielding is equal to $P_Y = SF(P) = 6.05$ kN. Since $D = 3h$, assume $M = M_P$ and calculate P_P. Determine the ratio P_P/P_Y.

   Ans. $P_P = 14.33$ kN, $P_P/P_Y = 2.37$

3. Let the steel in the curved beam in Example 8-4.1 have a flat top at the yield point stress $Y = 280$ MPa. Determine the fully plastic moment for the curved beam. Note that the original cross section must be used. The distortion of the cross section increases the fully plastic moment for a positive moment.

REFERENCES

1. S. Timoshenko and J. Goodier, *Theory of Elasticity*, 3rd Ed., McGraw-Hill, 1970.
2. H. Bleich, "Die Spannungsverteilung in den Gurtungen gekrümmter Stabe mit T- und I-formigem Querschnitt," *Der Stahlbau, Beilage zur Zeitschrift, Die Bautechnik*, Vol. 6, No. 1, Jan. 6, 1933, pp 3–6. English translation, Navy Department, The David W. Taylor Model Basin, Translation 228, January 1950.
3. C. G. Anderson, "Flexural Stresses in Curved Beams of I- and Box Sections," presented to the Institution of Mechanical Engineers, November 3, 1950.
4. T. von Kármán, *Zeitschrift des Vereines deutscher Ingenieure*, Vol. 55, 1911, p. 1889.
5. S. Timoshenko, "Bending Stresses in Curved Tubes of Rectangular Cross-Section," *Trans. ASME*, Vol. 45, 1923, pp. 135–140.
6. D. C. Broughton, M. E. Clark, and H. T. Corten, "Tests and Theory of Elastic Stresses in Curved Beams Having I- and T-Sections," *Experimental Mechanics*, Vol. 8, No. 1, 1950, pp. 143–155.
7. J. C. McWhorter, H. R. Wetenkamp, and O. M. Sidebottom, "Finite Deflections of Curved Beams," *Journal of the Engineering Mechanics Division, Proceedings of ASCE*, April 1971, pp. 345–358.

Additional References

1. Y. Enda, "Analysis of Thin-Walled Curved Beams by the Transfer Matrix Method," *Advances in Computational Methods in Structural Mechanics and Design*, University of Alabama press, Huntsville, 1972, pp. 757–774.
2. V. Z. Vlasov, *Theory of Thin-Walled Elastic Beams*, Gihodo Press, Tokyo, 1967.
3. F. Wansleben, "Die Berechnung drehfester gekrümmten dünnwandigen Trägern," *Der Stahlbau*, Vol. 33, 1964, pp. 364–372.

EXHIBIT 3

ENGINEERING DESIGN

FIFTH EDITION

George E. Dieter
University of Maryland

Linda C. Schmidt
University of Maryland

Mc Graw Hill
Connect Learn Succeed™

The McGraw·Hill Companies

Mc Graw Hill Connect Learn Succeed™

ENGINEERING DESIGN, FIFTH EDITION

Published by McGraw-Hill, a business unit of The McGraw-Hill Companies, Inc., 1221 Avenue of the Americas, New York, NY 10020.

Some ancillaries, including electronic and print components, may not be available to customers outside the United States.

This book is printed on acid-free paper.

1 2 3 4 5 6 7 8 9 0 DOC/DOC 1 0 9 8 7 6 5 4 3 2

ISBN 978-0-07-339814-3
MHID 0-07-339814-4

Vice President & Editor-in-Chief: *Marty Lange*
Vice President of Specialized Publishing: *Janice M. Roerig-Blong*
Editorial Director: *Michael Lange*
Global Publisher: *Raghothaman Srinivasan*
Senior Sponsoring Editor: *Bill Stenquist*
Marketing Manager: *Curt Reynolds*
Developmental Editor: *Lorraine K. Buczek*
Project Manager: *Melissa M. Leick*
Design Coordinator: *Brenda A. Rolwes*
Cover Designer: *Studio Montage, St. Louis, Missouri*
Cover Image: © *Getty Images RF*
Buyer: *Laura Fuller*
Media Project Manager: *Balaji Sundararaman*
Compositor: *Cenveo Publisher Services*
Typeface: *10.5/12 Times Roman*
Printer: *R. R. Donnelley*

Library of Congress Cataloging-in-Publication Data

Dieter, George Ellwood.
Engineering design. — 5th ed. / George E . Dieter, Linda C . Schmidt.
p. cm.
ISBN 978-0-07-339814-3 (acid-free paper)
1. Engineering design. I. Schmidt, Linda C. II. Title.
TA174.D495 2013
620'.0042—dc23

2011048958

www.mhhe.com

1.3.3 A Problem-Solving Methodology

Designing can be approached as a problem to be solved. A problem-solving methodology that is useful in design consists of the following steps.[1]

- Definition of the problem
- Gathering of information
- Generation of alternative solutions
- Evaluation of alternatives and decision making
- Communication of the results

This problem-solving method can be used at any point in the design process, whether at the conception of a product or the design of a component.

Definition of the Problem

The most critical step in the solution of a problem is the *problem definition* or formulation. The true problem is not always what it seems at first glance. Because this step seemingly requires such a small part of the total time to reach a solution, its importance is often overlooked. Figure 1.4 illustrates how the final design can differ greatly depending upon how the problem is defined.

The formulation of the problem should start by writing down a problem statement. This document should express as specifically as possible what the problem is. It should include objectives and goals, the current state of affairs and the desired state, any constraints placed on solution of the problem, and the definition of any special technical terms. The problem-definition step in a design project is covered in detail in Chap. 3.

Problem definition often is called *needs analysis*. While it is important to identify the needs clearly at the beginning of a design process, it should be understood that this is difficult to do for all but the most routine design. It is the nature of the design process that new needs are established as the design process proceeds because new problems arise as the design evolves. At this point, the analogy of design as problem solving is less fitting. Design is problem solving only when all needs and potential issues with alternatives are known. Of course, if these additional needs require reworking those parts of the design that have been completed, then penalties are incurred in terms of cost and project schedule. Experience is one of the best remedies for this aspect of designing, but modern computer-based design tools help ameliorate the effects of inexperience.

Gathering Information

Perhaps the greatest frustration you will encounter when you embark on your first design project will be either the dearth or the plethora of information. Your assigned problem may be in a technical area in which you have no previous

1. A similar process called the guided iteration methodology has been proposed by J. R. Dixon; see J. R. Dixon and C. Poli, *Engineering Design and Design for Manufacturing*, Field Stone Publishers, Conway, MA, 1995. A different but very similar problem-solving approach using TQM tools is given in Sec. 4.6.

EXHIBIT 4

UNITED STATES DISTRICT COURT

DISTRICT OF ARIZONA

In Re Bard IVC Filters Liability Litigation	) No. MD-15-02641-PHX-DGC

DO NOT DISCLOSE - SUBJECT TO FURTHER CONFIDENTIALITY REVIEW

VIDEOTAPED DEPOSITION OF PAUL L. BRIANT, Ph.D., P.E.

THURSDAY, JULY 13, 2017

BURLINGAME, CALIFORNIA

Reported by: Shelley M. Sailor, CSR No. 10254

GOLKOW LITIGATION SERVICES

877.370.3377 ph | 917.591.5672 fax

deps@golkow.com

Q. Without tilt during normal breathing, what is the maximum strain you believe these filters will encounter?

A. So I don't think we could answer that with -- necessarily with the work that we've done. The work that I've done has been to understand, assess the calculations by Dr. McMeeking. To answer the question that you are asking about would require an assessment of the maximum or foreseeable motion of the IVC as well as -- and beyond that, really, I'm not sure that that -- that would be a very hard question to answer just due to the complicated nature of the IVC. So it require more than just that motion. It would be -- you know, it's a very complex environment.

Q. In a complex environment, such as the vena cava, is it important to run your calculations to account for a worst case scenario?

A. One should try and estimate the foreseeable worst case or maximum conditions that their device is expected to undergo.

Q. Why?

A. To understand, assess the fatigue performance of the device.

Q. Why would you want to do that under the

maximum stress scenario? Why is that important to do as an engineer?

A. Again, to try and understand and evaluate the fatigue performance to try to minimize complications such as fatigue fracture of the device.

Q. Do you have any opinion as to the foreseeable motion of the IVC, the inferior vena cava?

A. Do I have an opinion on what that is?

Q. Yes.

A. No.

Q. Did you see anywhere in Dr. McMeeking's report where he has an opinion on that topic?

A. So he lays out sort of three different conditions or scenarios where -- that he analyzed the device under and which are the three pulsations we used. Namely, 1 millimeter 18 percent and 50 percent. And he bases those off of literature. We have, or I have seen, through my review of the literature, pulsations on the order of all of those. Some of that being relatively recent literature.

Q. Are you critical of his use of those values for the pulsation of the IVC?

A. I am not critical of his use of those

values. Where I am critical is of the assumptions and the methodology used with those assumptions to calculate the strains under those pulsations.

Q. You keep saying "methodology used with those assumptions," and I'm having trouble understanding what the difference between your use of criticizing his methodology versus his assumptions.

A. Sure. So what I mean by that is when one makes assumptions in an analysis -- let me back up.

Part of the methodology of performing analysis like this is evaluating or justifying assumptions that are made. And Dr. McMeeking makes certain assumptions about the motion of the IVC after the filter has been implanted, and I haven't seen justification for those assumptions.

Q. So that's what I understood your disagreement to be is with several of the assumptions he makes in applying the FEA process.

A. Correct.

Q. But the FEA process is a valid process to use.

A. I mean, overall, yes. Using FEA to calculate strains in a device is certainly fine.

Q. It's what you use.

A. Absolutely.

Q. And it's a well-recognized method of calculating strains within the engineering community.

A. Certainly. And analytical methods are also used. Again, where it comes down to is the assumptions in performing those calculations and justifying those assumptions.

Q. If you make improper assumptions even using a valid methodology like FEA, you will get incorrect or unreliable results.

A. You could, yes.

Q. And that's what you believe happened here, is there were incorrect assumptions that aren't justified that result in what I said would be unreliable results.

A. Yes. They're overly conservative.

Q. But you have no criticisms of his attempt to use the FEA methodology to conduct the cyclic strain analysis.

A. No.

Q. Am I correct?

A. Correct.

Q. So then let me go to his report before I go through your configurations 1 through 3. It would

EXHIBIT 5

Paul Briant, Ph.D., P.E.

IN THE CIRCUIT COURT OF THE 17TH JUDICIAL CIRCUIT, IN
AND FOR BROWARD COUNTY, FLORIDA

CLARE AUSTIN,

Case No.

Plaintiff, CACE-15-008373 Div:07

v.

C.R. BARD, INC., a foreign corporation, and BARD PERIPHERAL VASCULAR, INC., an Arizona corporation, MATTHEW ROBBINS, M.D., and CLEVELAND CLINIC FLORIDA,

Defendants.

________________________________/

VIDEOTAPED DEPOSITION OF
PAUL BRIANT, Ph.D., P.E.
Held at San Francisco Airport Marriott Waterfront
1800 Old Bayshore Highway
Burlingame, California 94010
Friday, August 12, 2016, 8:56 a.m.

Reported by:
MARY ANN SCANLAN, CSR No. 8875, RMR, CRR, CCRR, CLR

environment it's going to be in?

A. If you have a set of conditions that you would expect, one could analyze the response of the device under those conditions.

Q. Now, you say that companies will come to you and ask you to do an engineering analysis of a new device.

A. Correct.

Q. One that isn't on the market yet.

A. Correct.

Q. Have you ever done that for an inferior vena cava filter?

A. Myself, no, I have not done that for an inferior vena cava filter.

Q. Has anybody at Exponent?

A. Yes.

Q. Who?

A. There is a colleague of mine named Ming Wu, who I know did one. And Brad James may have. He's another engineer at Exponent.

Q. For what company?

A. To be honest, I don't remember which companies they were. I know they were not Bard, though.

Q. As an engineer, when you analyze a new

medical device, do you agree that it should be evaluated from the worst case scenario?

A. One would try to come up with the foreseeable worst case conditions that the device would see, based on available information.

Q. And the reason the worst case scenario is analyzed is eventually for the protection of the patient?

A. Correct. To -- and broadly in general, not just medical devices, to try to prevent failure.

Q. Safety is a factor that a worst case scenario is applied by an engineer?

A. One tries to ensure that a device will not break or fail by estimating the worst case conditions a device would be expected to see.

Q. And an engineer who engages in a worst case scenario analysis has to be objective and provide the accurate information to his or her client that has retained him.

A. Correct.

Q. You obtained your Ph.D. from Stanford University in 2008?

A. Yes.

Q. Let's start from the beginning. You obtained a bachelor of science in mechanical

Q. Thank you.
A. Yes.
Q. Have we covered all the issues relating to number one?
A. I believe we have.
Q. When you talk about conservative, you're talking -- Dr. McMeeking is considering worst case scenarios, true?
A. I think that what he has, what he has applied is overly conservative as opposed to a realistic worst case.
Q. But we can agree on a rule that an engineer should consider worst case scenarios in evaluating and accessing any type of medical device that's going to be implanted in a vessel such as the vena cava, correct?
A. I would agree that you would try to analyze your device under worst case scenarios that are supported by information that you have.
Q. Meaning that you have to be overly -- you have to be conservative?
A. You have to be conservative, yes.
Q. Being conservative is being a reasonable engineer?
A. I would say that being conservative is

important in device design, yes.
Q. A reasonable engineer should be conservative?
A. I would say that -- yeah, engineers should be conservative.
Q. Thank you.
Number two, are we ready to move to number two?
A. We are.
Q. The nitinol fatigue strength implied by Dr. McMeeking is overly conservative. Now, I take it that number two does not apply to his new calculations?
A. No.
Q. Do you agree?
A. That's correct; neither does number one.
Q. So number one and number two can be removed, based upon Dr. McMeeking's new calculations?
A. Oh, sorry. They don't apply to the new calculations that I have seen because the new calculations involve tilting and, to some degree, perforation, while these are related to fatigue.
Q. Number three, rigid boundary condition applied to the point where the struts end of the cap

A. Potentially nothing; it depends how much they go up, but --

Q. Is that a consideration that needs to be accounted for when a manufacturer is assessing whether a filter made of nitinol will have a tendency to perforate?

A. I think it's -- I think it's something that one should think about.

Q. When a manufacturer of an IVC filter does bench testing, you agree the manufacturer should consider the worst case scenarios, correct?

A. The manufacturers should do research to understand the worst case conditions that it might see.

Q. And a list of reasonable worst case conditions that the G2 should be able to survive after implantation?

A. Have I come up with that list?

Q. Well, the design and the experimentation of the best changes should look at whether it can survive after implantation and not perforate, true?

A. I'm sorry. Can you repeat that?

Q. Sure.

A worst case scenario that should be considered by the manufacturer is does it have a

tendency to perforate?
A. I think perforation should be looked at in terms of whether it's possible.
Q. And find out what the root cause of perforation is and work on eliminating that?
A. I think they should investigate it, yes.
Q. What about endothelialization?
A. Endothelialization, I think, is likely going to happen in some cases in some people, pretty much regardless.
Q. Should that be considered in terms of to what extent it can be reduced or eliminated as a worst case scenario?
A. Again, I think, you know, within the desired design goals or required design goals of the product, that a manufacturer should think about it or consider it.
Q. So you agree?
A. Can you repeat the question?
MR. O'CONNOR: Sure. Can you read it back?
THE REPORTER:
(Record read as follows:
"Question: Should that be considered in terms of to what

Q. Can it lead to the filter not remaining centered?
A. I haven't done any analysis, so I'm not sure.
Q. And you agree Dr. McMeeking has analyzed those issues?
A. He has done some analysis on it.
Q. And he has determined that perforation and endothelialization do affect tilting and do affect additional perforation?
A. I would like to review his calculations. I haven't looked at those yet.
Q. But you agree those are things that he did address?
A. My understanding from reading his deposition is that he has -- I'm not sure about endothelialization, but he has started to look at perforation and its relationship with tilt.
Q. And I think we've talked about this before: A manufacturer should consider the worst case scenario when assessing expansion and contraction of the vena cava in relation to filter implantation.
A. In -- I think a manufacturer should analyze the devices under expected worst case

conditions.
Q. Which would include expansion and contraction of the vena cava?
A. Yes.
Q. And the manufacturer should consider worst case conditions that the G2 may encounter after implantation, including propensity to tilt?
A. Should the manufacturer consider -- I'm just repeating the question -- should the manufacturer consider worst case -- I'm sorry. Do you mind reading it back?
Q. I can say it again.
A manufacturer should consider tilting in evaluating worst case scenarios in how the G2 responds after implantation?
A. Again, yes, I think that that should be -- if that is a known thing to occur, then that should be looked at in terms of what it can do.
Q. If it is known that legs of the G2 perforate walls of the vena cava, can that lead to tilting?
A. I haven't done any analysis.
Q. But I think you told me before it sounds reasonable?
A. It's not unreasonable.

EXHIBIT 6
(Filed Under Seal)

EXHIBIT 7

IN THE UNITED STATES DISTRICT COURT

NORTHERN DISTRICT OF GEORGIA

ATLANTA DIVISION

-----------------------------------x

PAMELA B. CASON and KERRY B. CASON,

Plaintiffs,

vs. Civil Action No.: 1:12-cv-1288

C.R. BARD, INC. and BARD PERIPHERAL VASCULAR, INC.,

Defendants.

-----------------------------------x

Deposition of CHRISTINE L. BRAUER, Ph.D.

Washington, D.C.

Friday, May 23, 2014

9:30 a.m.

Reported by: Amy E. Sikora-Trapp, RPR, CRR,

Former CSR-NY, CLR

I'm asking and she knows exactly what she's doing. She's changing the question to satisfy the answer she wants to give me because she doesn't want to answer my question.

Q. I'm going to ask the court reporter to read the question again, and I want you to answer my question.

A. I'll do my best.

Q. No, you're not. I mean, you've been saying that all day. You're not doing your best.

(Record read.)

THE WITNESS: One more time, please.

(Record read.)

MR. LOPEZ: Oh, my goodness.

A. If, hypothetically speaking, the Recovery filter was shown to no longer be substantially equivalent to its predicate device, hypothetically speaking, then hypothetically speaking it would not be an appropriate predicate device for G2.

Q. Thank you. I knew you could do